Brion's Theorem and q-Catalan numbers

Séminaire Lotharingien de Combinatoire 94, Bad Boll

Drew Armstrong

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University of Miami and RUB Bochum
www.math.miami.edu/~armstrong

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And the rational q-Catalan number is defined as

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Facts

1. $\mathrm{Cat}(a,b)\in\mathbb{N}$ (this is not difficult)
2. $\mathrm{Cat}(a,b)_q\in\mathbb{Z}[q]$ (this is not difficult)
3. $\mathrm{Cat}(a,b)_q\in\mathbb{N}[q]$ (this is difficult)

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Stanton's Problem (~2005): Find a natural set X with $\#X = \mathrm{Cat}(a, b)$ and a function stat $: X \to \mathbb{N}$ satisfying

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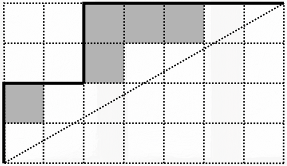
One solution to this problem is known. Let $\mathrm{Dyck}_{a,b}$ be the set of Dyck paths in an $a \times b$ rectangle, though of as a sequence of a up steps (u) and b right steps (r) staying above the diagonal. For example, uurruurrrrr $\in \mathrm{Dyck}_{4,7}$.



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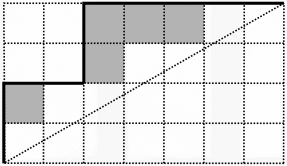

Given $P \in \mathrm{Dyck}_{a,b}$ let area(P) be the number of full squares between the path and the diagonal. For example, $\mathrm{area}(\mathrm{uurruurrrrr}) = 5$.

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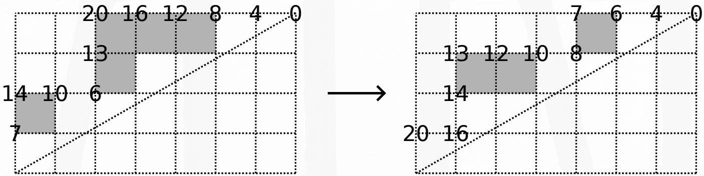
Define the sweep map $\mathrm{Dyck}_{a,b} \to \mathrm{Dyck}_{a,b}$ as follows. Label each step with endpoint (x, y) by the "height" $yb - ax$. Then sort the steps by decreasing labels.



u	u	r	r	u	u	r	r	r	r	r
7	14	10	6	13	20	16	12	8	4	0

u	r	u	u	r	r	r	u	r	r	r
20	16	14	13	12	10	8	7	6	4	0

Example: sweep(uurruurrrrr) = uruurrrurrr.

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Theorem (conjectured by many, proved by Mellit 2021)

$$\mathrm{Cat}(a, b)_q = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P) - \mathrm{area}(\mathrm{sweep}(P)) + (a-1)(b-1)/2}$$

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Actually, Mellit proved that the statistics area and area ∘ sweep give the rational q,t-Catalan number

$$\mathrm{Cat}(a,b)_{q,t} = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathsf{area}(P)} t^{\mathsf{area}(\mathsf{sweep}(P))}.$$

2. Lattice points

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This is not a satisfactory resolution of Stanton's problem because:

- The function $\mathrm{stat}(P) = \mathrm{area}(P) - \mathrm{area}(\mathrm{sweep}(P)) + (a-1)(b-1)/2$ is difficult to work with.
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Monotonicity Conjecture

For $a, b, c \geq 1$ with $\gcd(a, b) = \gcd(a, c) = 1$ and $b < c$ we have

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Example:

$$\begin{aligned} \mathrm{Cat}(3,1)_q &= 1, \\ \mathrm{Cat}(3,2)_q &= 1 + q^2, \\ \mathrm{Cat}(3,4)_q &= 1 + q^2 + q^3 + q^4 + q^6, \\ \mathrm{Cat}(3,5)_q &= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8. \end{aligned}$$

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In this talk I will present a new point of view that I have used to prove monotonicity for the infinite set of triples a, b, c satisfying $a \leq 20$. Here's the idea:

~~Dyck paths~~ $\rightsquigarrow$ lattice points.

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Let $\mathcal{L} := \mathbb{Z}^{a-1}$ be the weight lattice with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}^{-1} \mathbf{y}$$

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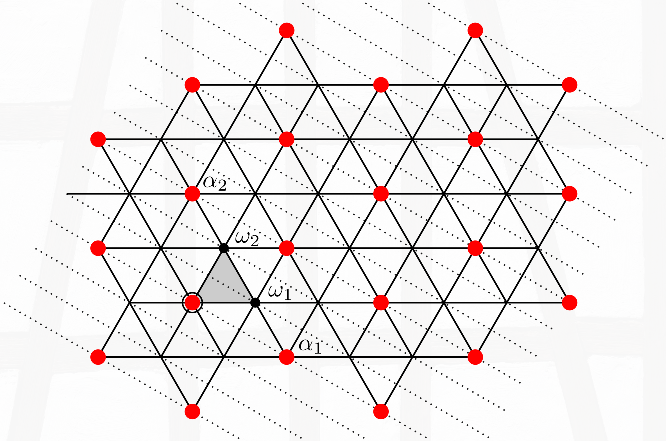
The coordinate basis are called the fundamental weights $\omega_1, \ldots, \omega_{a-1}$ and the dual basis are called the simple roots $\alpha_1, \ldots, \alpha_{a-1}$.

$$\begin{aligned} \omega_1 &= (1, 0, 0, \ldots, 0) & \alpha_1 &= (2, -1, 0, \ldots, 0, 0) \\ \omega_2 &= (0, 1, 0, \ldots, 0) & \alpha_2 &= (-1, 2, -1, \ldots, 0) \\ &\vdots & &\vdots \\ \omega_{a-1} &= (0, 0, \ldots, 0, 1) & \alpha_{a-1} &= (0, 0, \ldots, 0, -1, 2) \end{aligned}$$

Example $a = 3$**:** The red points are the root lattice

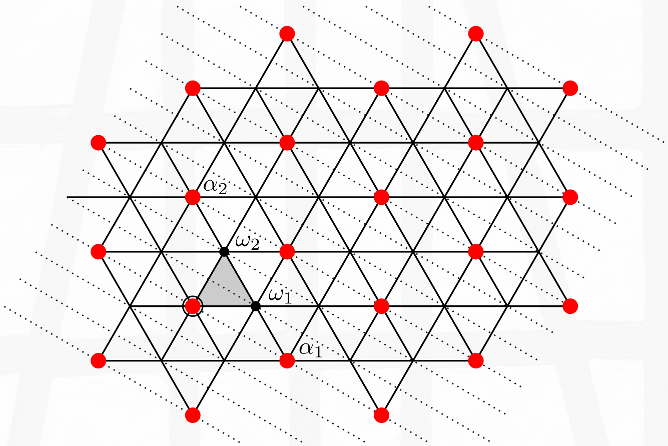
$$\mathcal{R} = \mathbb{Z}\{\alpha_1, \ldots, \alpha_{a-1}\}.$$



2. Lattice points

Example $a = 3$**:** The grey triangle Δ is the fundamental alcove

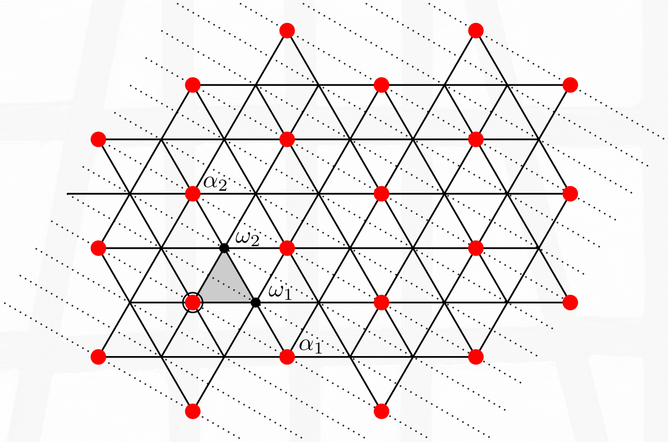
$$\Delta = \text{hull}\{0, \omega_1, \dots, \omega_{a-1}\}.$$



2. Lattice points

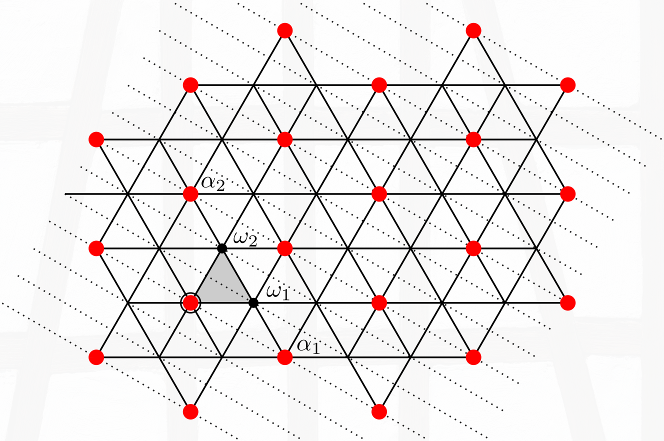
Example $a = 3$**:** And the dotted lines are the level sets of the tilted height function

$$T(x_1, x_2, \ldots, x_{a-1}) := x_1 + 2x_2 + \cdots + (a-1)x_{a-1} = a\langle \mathbf{x}, \omega_{a-1}\rangle.$$



Example $a = 3$**:** Note that we can also write

$$\mathcal{R} = \{\mathbf{x} \in \mathcal{L} : \color{red}{\mathrm{T}(\mathbf{x}) \equiv 0 \bmod a}\}.$$



Important Observation: For any integer b, the tilted height generating function for the dilated alcove $b\Delta$ is the q-binomial coefficient:

$$\sum_{\mathbf{x} \in (\mathcal{L} \cap b\Delta)} q^{\mathrm{T}(\mathbf{x})} = \begin{bmatrix} a-1+b \\ a-1 \end{bmatrix}_q .$$

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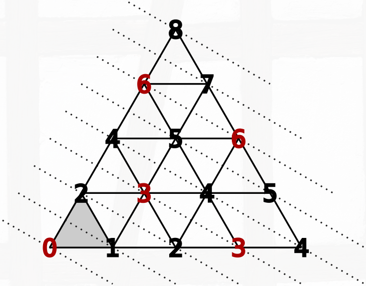

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8$$

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But if $\gcd(a, b) = 1$ then Mark Haiman tells us that the number of points of the root lattice in the dilated alcove $b\Delta$ is the rational Catalan number:

$$\#(\mathcal{R} \cap b\Delta) = \mathrm{Cat}(a, b) = \frac{1}{a}\binom{a-1+b}{a-1}.$$

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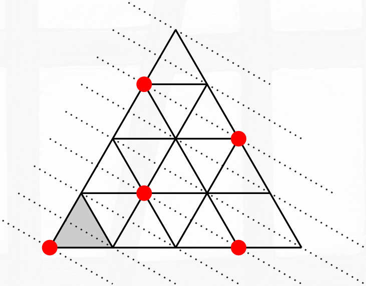
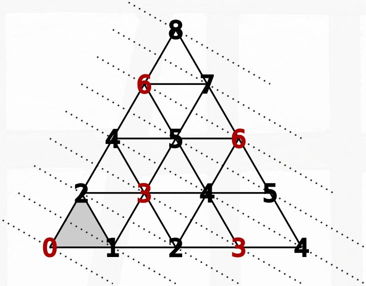
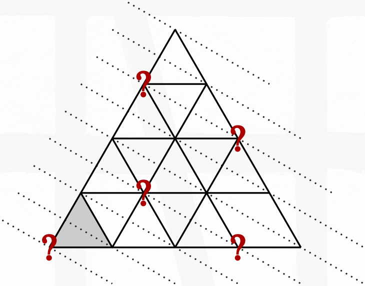


$$\mathrm{Cat}(3,4)=\frac{1}{3}\binom{6}{2}=5$$

We need some way to “divide by $[a]_q$” using the geometry.





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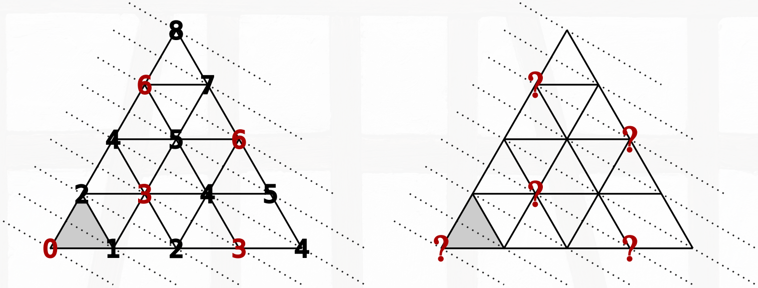
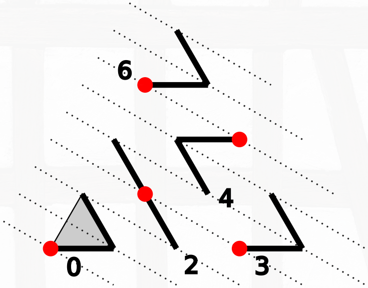


$$\mathrm{Cat}(3,4)_q = \frac{1}{[3]_q}\begin{bmatrix}6\\2\end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6 = \sum_{\mathbf{x}\in(\mathcal{R}\cap 4\Delta)} q^?$$

3. Johnson statistics

Preview:



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Definition

A function $\mathrm{J} : \mathcal{R} \to \mathbb{Z}$ is called a Johnson statistic if it satisfies:

- (Periodic) For all $\mathbf{x} \in \mathcal{R}$ and $\mathbf{y} \in \mathcal{L}$ we have $\mathrm{J}(\mathbf{x} + a\mathbf{y}) = \mathrm{J}(\mathbf{x}) + a\mathrm{T}(\mathbf{y})$.

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- (Catalan) For all $\gcd(a, b) = 1$ we have

$$\sum_{\mathbf{x} \in (\mathcal{R} \cap b\Delta)} q^{\mathsf{J}(\mathbf{x})} = \mathrm{Cat}(a, b)_q.$$

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Conjecture

Johnson statistics exist for all a.

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Theorem

Johnson statistics exist for all $a \leq 20$.

3. Johnson statistics

Because of periodicity a Johnson statistic is determined by its values on the cosets $\mathcal{R}/a\mathcal{L}$. We note that $\#(\mathcal{R}/a\mathcal{L}) = a^{a-2}$ because

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There is a canonical set of coset representatives. Define the fundamental box

$$\mathrm{Box} = \{\mathbf{x} \in \mathbb{R}^{a-1} : 0 \leq x_i < a \text{ for all } i\}.$$

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Then taking remainders mod a gives a bijection

$$\mathcal{R}/a\mathcal{L} \quad \longleftrightarrow \quad \mathcal{R} \cap \mathrm{Box}.$$

We have the following expressions on the fundamental box.

Theorem

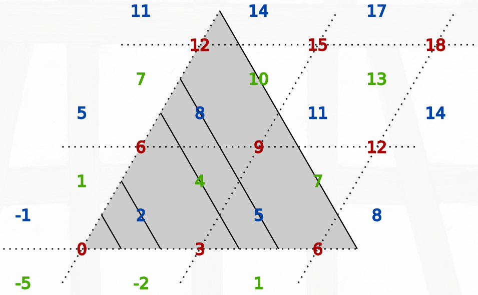
If $\mathsf{J} : \mathcal{R} \to \mathbb{Z}$ is a Johnson statistic then we have

$$[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}} = \sum_{\mathbf{x}\in(\mathcal{R}\cap\mathrm{Box})} q^{\mathsf{J}(\mathbf{x})},$$

$$\mathrm{Cat}(a,b)_q = \sum_{\mathbf{x}\in(\mathcal{R}\cap\mathrm{Box})} q^{\mathsf{J}(\mathbf{x})} \begin{bmatrix} a-1+\lfloor (b-\sum_i x_i)/a \rfloor \\ a-1 \end{bmatrix}_{q^a}.$$

3. Johnson statistics

When $a = 3$ there is exists a unique Johnson statistic.



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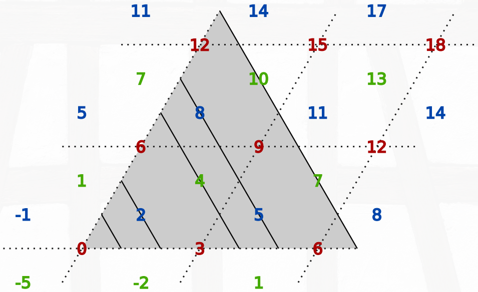

Monotonicity:

$$\mathrm{Cat}(3,1)_q = 1,$$

$$\mathrm{Cat}(3,2)_q = 1 + q^2,$$

$$\mathrm{Cat}(3,4)_q = 1 + q^2 + q^3 + q^4 + q^6,$$

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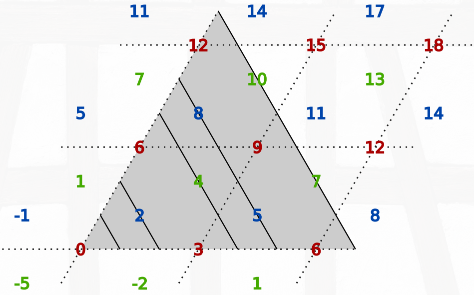

Since $a^{a-2} = 3^1$ the fundamental domain contains three points:

$$\mathcal{R} \cap \mathrm{Box} = \{(x_1, x_2) : 0 \le x_i < 3 \text{ and } x_1 + 2x_2 \equiv 0 \bmod 3\}$$
$$= \{(0,0), (1,1), (2,2)\}.$$

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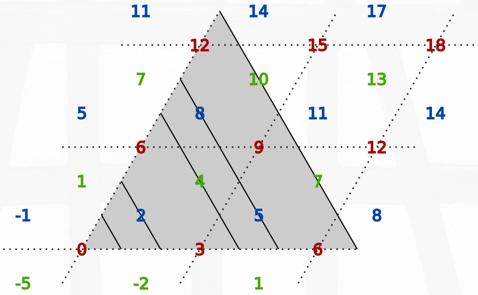

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$$\sum_{\mathbf{x}\in(\mathcal{R}\cap\mathrm{Box})} q^{\mathsf{J}(\mathbf{x})} = q^0 + q^2 + q^4 = [3]_{q^2}.$$

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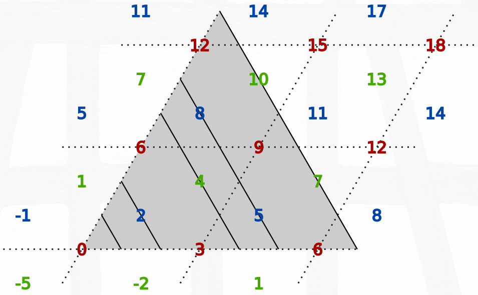
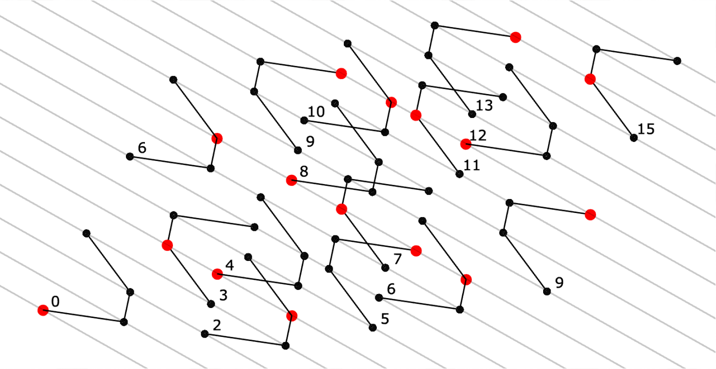


Since $a^{a-2} = 3^1$ the fundamental domain contains three points:

$$\mathrm{Cat}(3,b)_q = q^0 \begin{bmatrix} 2 + \lfloor \frac{b}{3} \rfloor \\ 2 \end{bmatrix}_{q^3} + q^2 \begin{bmatrix} 2 + \lfloor \frac{b-2}{3} \rfloor \\ 2 \end{bmatrix}_{q^3} + q^4 \begin{bmatrix} 2 + \lfloor \frac{b-4}{3} \rfloor \\ 2 \end{bmatrix}_{q^3}.$$

3. Johnson statistics

When $a = 4$ there are $a^{a-2} = 4^2 = 16$ cosets. Here is a Johnson statistic.



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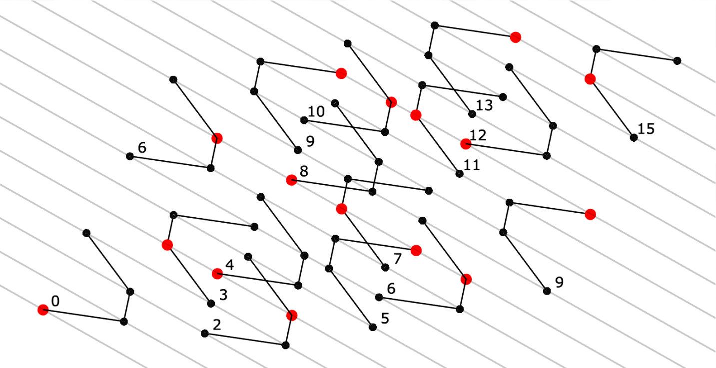

Don't ask me any questions about this picture. For $a \geq 5$ I do not know any natural construction of Johnson statistics, but I know that they exist for $a \leq 20$.

4. Brion's theorem and rational q-Catalan numbers

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Given a set $S \subseteq \mathbb{R}^{a-1}$ define the Johnson generating function:

$$(S)_{\mathsf{J}} = \sum_{\mathbf{x} \in (\mathcal{R} \cap S)} q^{\mathsf{J}(\mathbf{x})}.$$

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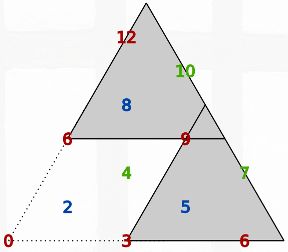
The formula $(\mathrm{Box})_J = [a]_{q^2}[a]_{q^3} \cdots [a]_{q^{a-1}}$ is proved by inclusion-exclusion, e.g.,



$$\mathrm{Cat}(3,7)_q - q^3\mathrm{Cat}(3,4)_q - q^6\mathrm{Cat}(3,4)_q + q^9\mathrm{Cat}(3,1)_q = [3]_{q^2}$$

To prove this we need a q-identity.

Lemma

Define the Pochhammer symbol $(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1})$. For any set $J \subseteq \mathbb{N}$ we let $\sum J$ denote the sum $\sum_{j\in J} j$. Then for any u we have

$$\sum_{J\subseteq\{1,\ldots,a-1\}} (-1)^{\#J} q^{a\sum J} \frac{(uq^{-a\#J};q)_{a-1}}{(q;q)_{a-1}} = \frac{(q^a;q^a)_{a-1}}{(q;q)_{a-1}},$$

Note that the right hand side is independent of u.

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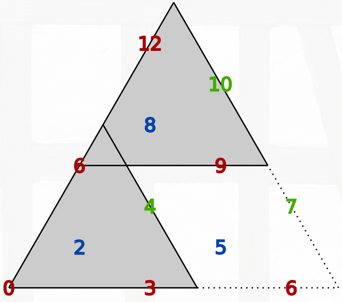
Note that the right hand side is independent of u.

We put $u = q^{b+1}$ for some large b with $\gcd(a,b) = 1$ and divide by $[a]_q$ to get

$$\begin{aligned}(\mathrm{Box})_{\mathrm{J}} &= \sum_{J\subseteq\{1,\dots,a-1\}} (-1)^{\#} q^{a\sum J} \mathrm{Cat}(a, b-a\#J)_q \\ &= [a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}.\end{aligned}$$

4. Brion's theorem and rational q-Catalan numbers

A similar identity gives a generating function for a fundamental domain at each vertex of the dilated simplex $b\Delta$:



$$\mathrm{Cat}(3,7)_q - \mathrm{Cat}(3,4)_q - q^6\mathrm{Cat}(3,4)_q + q^6\mathrm{Cat}(3,1)_q = q^6[3]_q$$

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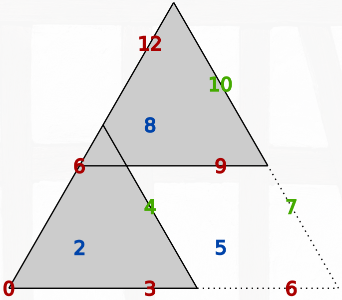

If $\gcd(a,b)=1$ then the kth vertex of $b\Delta$ has the following q-analogue of a^{a-2}:

$$(\mathrm{Box}_b^k)_J = q^{bk-(a-1)\binom{k+2}{2}} \frac{1}{[a]_q} \prod_{i=1}^{k} [a]_{q^i} \prod_{i=1}^{a-1-k} [a]_{q^i}.$$

4. Brion's theorem and rational q-Catalan numbers

Let K_b^k be the vertex cone of $b\Delta$ at the kth vertex $b\omega_k$. By using the periodicity of Johnson statistics and the previous result, we obtain the Johnson generating function of the cone:

$$(K_b^k)_{\mathrm{J}} = \frac{(\mathrm{Box}_b^k)_{\mathrm{J}}}{\prod_{i \in \{0,1,\ldots,a-1\} \setminus \{k\}} (1 - q^{a(i-k)})} = \frac{1}{[a]_q} \frac{(-1)^k q^{bk + \binom{k+1}{2}}}{(q;q)_k (q;q)_{a-1-k}}.$$

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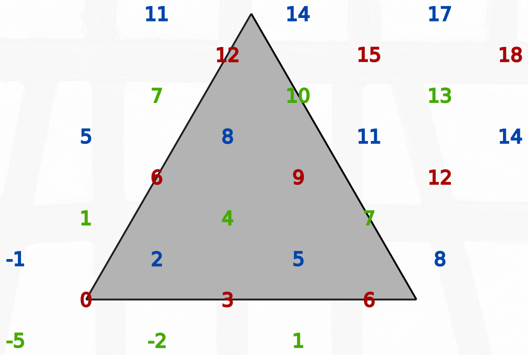
Finally, by applying the q-binomial theorem we obtain the following identity:

$$\boxed{(b\Delta)_J = \mathrm{Cat}(a,b)_q = \sum_{k=0}^{a-1} (K_b^k)_J.}$$

This is some q-analogue of Brion's formula for the dilated simplex $b\Delta$, which is a rational polytope with regard to the root lattice $\mathcal{R}$.

Example:



$$\frac{1+q^2+q^4}{(1-q^3)(1-q^6)}+\frac{q^5+q^6+q^7}{(1-q^{-3})(1-q^3)}+\frac{q^8+q^{10}+q^{12}}{(1-q^{-6})(1-q^{-3})}$$

$$=1+q^2+q^3+q^4+q^5+2q^6+q^7+q^8+q^9+q^{10}+q^{12}=\mathrm{Cat}(3,7)_q.$$

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- If we define the Johnson point enumerator

$$(S)_{\mathsf{J},\mathbf{z}} = \sum_{\mathbf{x}\in(\mathcal{R}\cap S)} q^{\mathsf{J}(\mathbf{x})}\mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x}\in(\mathcal{R}\cap S)} q^{\mathsf{J}(\mathbf{x})} z_1^{x_1}\cdots z_{a-1}^{x_{a-1}}$$

then the corresponding Brion's theorem holds when $q = 1$ or when $\mathbf{z} = (1, 1, \ldots, 1)$ but not in general.

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  then the corresponding Brion's theorem holds when $q = 1$ or when $\mathbf{z} = (1, 1, \ldots, 1)$ but not in general.
- Hence this does not follow from Chapoton-style q-Ehrhart theory.

Vielen Dank!

There is a secret in this picture.