Hyperplane Arrangements & Diagonal Harmonics

Drew Armstrong

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Coinvariants
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Theorems (Newton-Chevalley-etc):

- Let $\mathfrak{S}_n$ act on $S = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables.
Coinvariants

Theorems (Newton-Chevalley/etc):

- Let $\mathfrak{S}_n$ act on $\mathcal{S} = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables.
- Then we have

$$\mathcal{S}^{\mathfrak{S}_n} \cong \mathbb{C}[p_1, \ldots, p_n]$$

where $p_k = \sum_{i=1}^{n} x_i^k$ are the

**power sum symmetric polynomials.**
Coinvariants

Theorems (Newton-Chevalley-etc):

- The **coinvariant ring** $R := S/(p_1, \ldots, p_n)$ is isomorphic to the regular representation:

$$R \cong \mathbb{S}_n \otimes \mathbb{C}\mathbb{S}_n$$
Theorems (Newton-Chevalley-etc):

- The **coinvariant ring** \( R := S/(p_1, \ldots, p_n) \) is isomorphic to the regular representation:

\[
R \cong \mathfrak{S}_n \otimes \mathbb{C} \mathfrak{S}_n
\]

- And it’s **graded**, with **Hilbert series**

\[
\sum_i \dim R_i q^i = \prod_{j=1}^n (1 + q + \cdots + q^{j-1}) = [n]_q!
\]

“the q-factorial”
Coinvariants

What’s next?

- Let $\mathfrak{S}_n$ act on $DS = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ diagonally.
Coinvariants

What’s next?

- Let $\mathfrak{S}_n$ act on $DS = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ diagonally.

- (Weyl) Then the ring of diagonal invariants $DS\mathfrak{S}_n$ is generated by the “polarized” power sums

$$p_{k,\ell} = \sum_{i=1}^{n} x_i^k y_i^\ell \quad \text{for} \quad k + \ell > 0$$

NOT algebraically independent
Coinvariants

Hard Theorem (Haiman, 2001):

- The **diagonal coinvariant ring**

\[ DR := DS/(p_{k,\ell} : k + \ell > 0) \]

has dimension \((n + 1)^{n-1}\)
Coinvariants

Hard Theorem (Haiman, 2001):

• The **diagonal coinvariant ring**

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**Ongoing Project:**

• Describe the (bigraded) Hilbert/Frobenius series!
• New science of “parking functions”
Affine Permutations

- **Bijections**: \( \pi : \mathbb{Z} \rightarrow \mathbb{Z} \)

- **“Periodic”**: \( \forall k \in \mathbb{Z}, \pi(k + n) = \pi(k) + n \)

- **Frame of Reference**: \( \pi(1) + \pi(2) + \cdots + \pi(n) = \binom{n + 1}{2} \)
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**example**

\[
\begin{array}{ccccccccc}
  k & \cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
  \pi(k) & \cdots & -3 & -1 & 1 & 0 & 2 & 4 & 3 & 5 & 7 & \cdots \\
\end{array}
\]

The “window notation”: \( \pi = [0, 2, 4] \)
Affine Permutations

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The “window notation”: \( \pi = [0, 2, 4] \)

**Also observe:** \( \pi = \cdots (-3, -2)(0, 1)(3, 4)(6, 7) \cdots \)
Define affine transpositions:

\[(i, j)) := \prod_{k \in \mathbb{Z}} (i + kn, j + kn)\]
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\[ ((i, j)) := \prod_{k \in \mathbb{Z}} (i + kn, j + kn) \]

Then we have:

\[ \tilde{\mathcal{S}}_n = \langle ((1, 2)), ((2, 3)), \ldots, ((n, n + 1)) \rangle \]

“affine symmetric group” generated by “affine adjacent transpositions”
Affine Permutations

(Lusztig, 1983) says it’s a Weyl group

<table>
<thead>
<tr>
<th>“transposition”</th>
<th>“reflection in hyperplane”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2)$</td>
<td>$x_1 - x_2 = 0$</td>
</tr>
<tr>
<td>$(2, 3)$</td>
<td>$x_2 - x_3 = 0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$(n - 1, n)$</td>
<td>$x_{n-1} - x_n = 0$</td>
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Affine Permutations

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\vdots & \vdots \\
((n - 1, n)) & x_{n-1} - x_n = 0 \\
((n, n + 1)) & x_1 - x_n = 1 \\
\hline
\end{array}
\]

Abuse of notation:

\[\mathfrak{S}_n = \langle ((1, 2)), ((2, 3)), \ldots, ((n - 1, n)) \rangle\]

“finite symmetric group”
Picture of Affine $S_3$

Group elements = “alcoves”
Two ways to think
Two ways to think

Way 1.

\[ \tilde{\mathcal{S}}_n = \mathcal{S}_n \times \mathcal{S}_n \]

\[ = \text{(finite symmetric group)} \times \text{(minimal coset reps)} \]

\[ = \text{(which cone are you in?)} \times \text{(where in the cone?)} \]

\[ = \text{(permute window notation)} \times \text{(into increasing order)} \]
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Way 1.

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\[ = (\text{permute window notation}) \times (\text{into increasing order}) \]

example

\[ [6, -3, 8, -1] = [3, 1, 4, 2] \times [-3, -1, 8, 6] \]
For Posterity:
Note (finite) ascent sets in window notation

\[\begin{align*}
\triangle & = \emptyset \\
\vartriangle & = \{1\} \\
\vartriangle & = \{2\} \\
\blacktriangle & = \{1, 2\}
\end{align*}\]
What if we invert?
This is Way 2 to think.
\( \tilde{\mathcal{S}}_n = \mathcal{S}_n \times Q_n \)

\( \mathcal{S}_n \) semi-direct product with the root lattice

\( Q_n = \left\{ (r_1, \ldots, r_n) \in \mathbb{Z}^n : \sum_i r_i = 0 \right\} \)
This is Way 2 to think.

\[ \tilde{\mathcal{G}}_n = \mathcal{G}_n \rtimes Q_n \]
\[ \mathcal{G}_n \text{ semi-direct product with the root lattice} \]

\[ Q_n = \left\{ (r_1, \ldots, r_n) \in \mathbb{Z}^n : \sum_i r_i = 0 \right\} \]

In terms of window notation:

\[ [6, -3, 8, -1] = (2, 1, 4, 3) + 4 \cdot (1, -1, 1, -1) \]

"finite permutation + n times a root"

"division with remainder"
This is Way 2 to think.

A copy of $S_3$ around each root vector $\circ \in Q_3$.
Now for Shi and Ish
Consider a special simplex

Bounded by:

\[ x_1 - x_2 = -1 \]
\[ x_2 - x_3 = -1 \]
\[ \vdots \]
\[ x_{n-1} - x_n = -1 \]
\[ x_1 - x_n = 2 \]
Now for Shi and Ish

Consider a special simplex

It’s a dilation of the fundamental alcove by a factor of $n + 1$

Hence it contains $(n + 1)^{n-1}$ alcoves!
Now for Shi and Ish

Cut it with the **Shi arrangement**

Shi arrangement:

\[ x_i - x_j = 0, 1 \]

for all

\[ 1 \leq i < j \leq n \]
Now for Shi and Ish

And consider the "distance enumerator"
(called it "shi")
And consider the "distance enumerator" (call it "shi")

\[ \sum q^{\text{shi}} = 6 + 6q + 3q^2 + q^3 \]
Now for Shi and Ish

Next define a statistic on the root lattice:
Next define a statistic on the root lattice:

Given \( \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Q}_n \)

let \( j \) be maximal such that \( r_j \) is minimal.
Now for Shi and Ish

Next define a statistic on the root lattice:

Given $\mathbf{r} = (r_1, \ldots, r_n) \in Q_n$

let $j$ be maximal such that $r_j$ is minimal.

Then

$$\text{ish}(\mathbf{r}) := j - n(r_j + 1)$$
Next define a statistic on the root lattice:

Given $r = (r_1, \ldots, r_n) \in \mathbb{Q}_n$

let $j$ be maximal such that $r_j$ is minimal.

Then

$$ish(r) := j - n(r_j + 1)$$

$$ish(2, -2, 2, -2, 0) = 4 - 5 \cdot (-2 + 1) = 9$$

here $n = 5, \quad j = 4, \quad r_j = -2$
Now for Shi and Ish

ish spirals.
Now for Shi and Ish

**ish spirals.**

\[ \sum t^{ish} = 6 + 6t + 3t^2 + t^3 \]
Now for Shi and Ish

The joint distribution:

<table>
<thead>
<tr>
<th>shi</th>
<th>ish</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
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Now for Shi and Ish

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**Conjectures:**

- **Joint Symmetry:** \( \sum q^{shi} t^{ish} = \sum t^{shi} q^{ish} \)
Now for Shi and Ish

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Conjectures:

- Joint Symmetry: \( \sum q^{\text{shi}} t^{\text{ish}} = \sum t^{\text{shi}} q^{\text{ish}} \)
- In fact, we have \( \sum q^{\text{shi}} t^{\text{ish}} = \text{Hilbert series of } DR \text{ diagonal coinvariants} \)
Now for Shi and Ish

Finally...

to each alcove ascent set $\text{Asc}$ we associate the

**(Gessel) Fundamental Quasisymmetric Function**

$$F_{\text{Asc}} = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \text{Asc} \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}$$
Now for Shi and Ish

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\[ \begin{array}{ccc} 
\triangle & = & F_\emptyset \\
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$$\begin{array}{cccc}
\text{△} &=& F_\emptyset \\
\text{△} &=& F_{\{1\}} \\
\text{△} &=& F_{\{2\}} \\
\text{△} &=& F_{\{1,2\}} \\
\text{△} &=& \text{schur}(3) \quad \text{(trivial representation)} \\
\text{△} + \text{△} &=& \text{schur}(2,1) \quad \text{(the other one)} \\
\text{△} &=& \text{schur}(1,1,1) \quad \text{(sign representation)}
\end{array}$$
Now for Shi and Ish
Now for Shi and Ish

$$\sum q^{\text{shi} \cdot \text{ish}} F_{\text{Asc}} =$$

the "(q,t)-Catalan"
Now for Shi and Ish

(Shuffle) Conjecture:

\[ \sum q^{\text{shi} \cdot \text{ish}} F_{\text{Asc}} \text{ is the Frobenius series of } DR \]
Now for Shi and Ish

Theorem (me):

- My “shuffle conjecture” = The Shuffle Conjecture (HHLRU05).
Now for Shi and Ish

**Theorem (me):**

- My “shuffle conjecture” = The Shuffle Conjecture (HHLRU05).

- That is, ∃ (at least) two natural maps to parking functions:

  $$\begin{align*}
  \text{dinv} & \quad \text{shi} & \quad \text{area'} \\
  \text{area} & \quad \text{ish} & \quad \text{bounce}
  \end{align*}$$

  “Haglund-Haiman-Loehr statistics”
Where does it come from?
Where does it come from?

If you invert this picture…
Where does it come from?

...you will get this picture.
Where does it come from?

...you will get this picture.

The Shi Arrangement
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The Ish Arrangement

(please see Brendon’s talk)
Thanks!

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