This talk is an advertisement for....

- 1005.1949 (me)
- 1009.1655 (me and B.Rhoades)

Part I: **Ish**

The Affine Symmetric Group $\tilde{\mathfrak{S}}_n$

- Bijections: $\pi : \mathbb{Z} \to \mathbb{Z}$
- Periodic: $\forall k \in \mathbb{Z},\ \pi(k+n) = \pi(k) + n$
- Frame of Reference: $\pi(1) + \cdots + \pi(n) = \binom{n+1}{2}$

zB

$$\begin{array}{cc|ccc|ccc|ccc|c} k & \cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \pi(k) & \cdots & -3 & -1 & 1 & 0 & 2 & 4 & 3 & 5 & 7 & \cdots \end{array}$$

The “window notation”: $\pi = [0, 2, 4]$

Observe: $[0, 2, 4] = \cdots (-3, -2)(0, 1)(3, 4)(6, 7) \cdots$

The Affine Symmetric Group $\tilde{\mathfrak{S}}_n$

Define Affine Transpositions

$$((i,j)) := \prod_{k\in\mathbb{Z}} (i+kn, j+kn)$$

Then

$$\tilde{\mathfrak{S}}_n = \Big\langle ((1,2)), ((2,3)), \ldots, ((n,n+1)) \Big\rangle$$

"affine adjacent transpositions"

The Affine Symmetric Group $\tilde{\mathfrak{S}}_n$

(Lusztig, 1983) says it's a Weyl group.

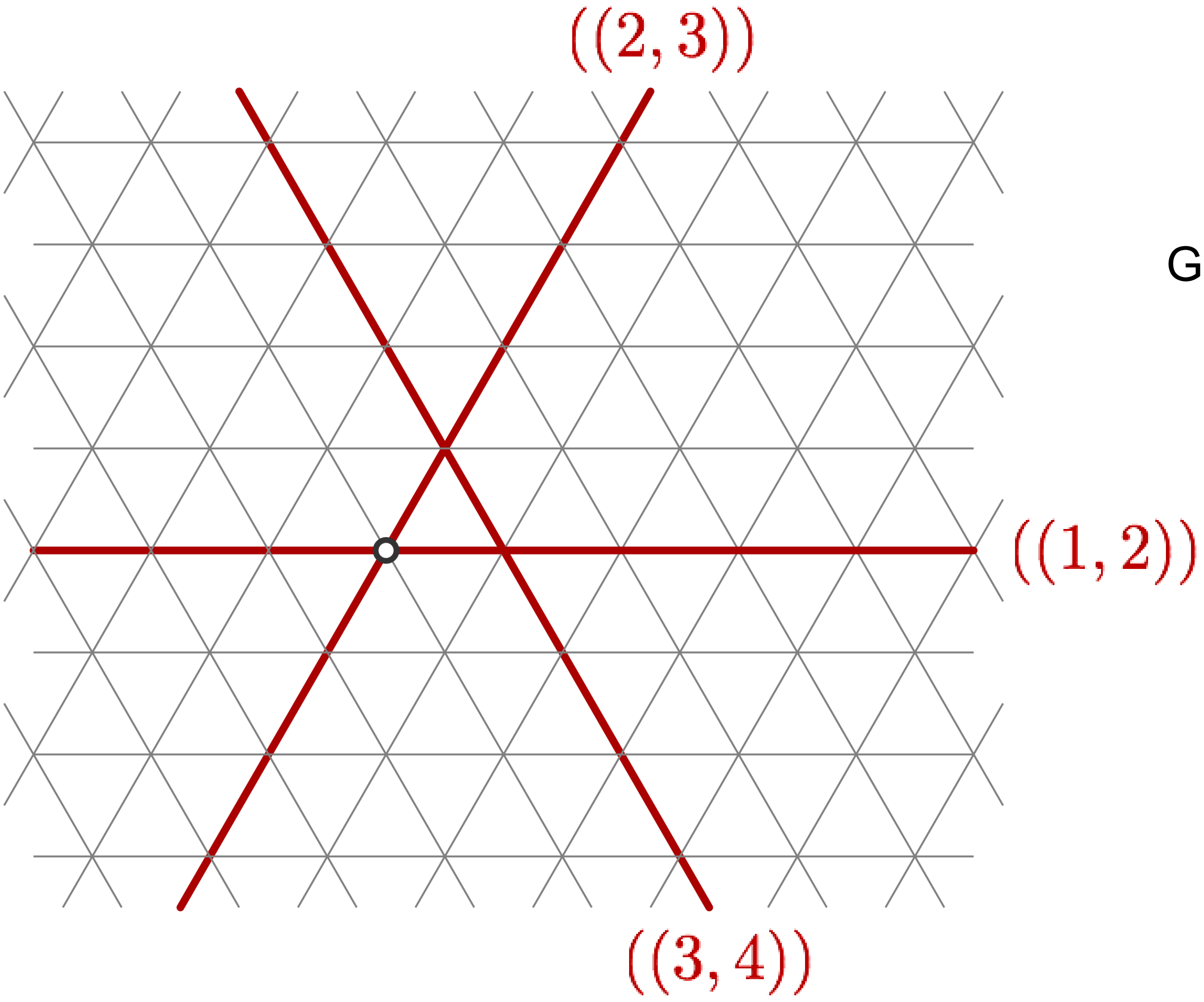
$$\begin{array}{ccc} \text{“transposition”} & & \text{“reflection in”} \\ ((1,2)) & \to & x_1 - x_2 = 0 \\ ((2,3)) & \to & x_2 - x_3 = 0 \\ & \vdots & \\ ((n-1,n)) & \to & x_{n-1} - x_n = 0 \\ ((n,n+1)) & \to & x_1 - x_n = 1 \end{array}$$

Abuse of Notation

$$\mathfrak{S}_n = \Big\langle ((1,2)), ((2,3)), \ldots, ((n-1,n)) \Big\rangle$$

“finite symmetric group”

Here's the first picture (of $\tilde{\mathfrak{S}}_3$) of the talk.



Group Elements
||
Alcoves

There are two ways to think.

1. $\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \times \mathfrak{S}^n$

$=$ (finite symmetric group) X (minimal coset reps)

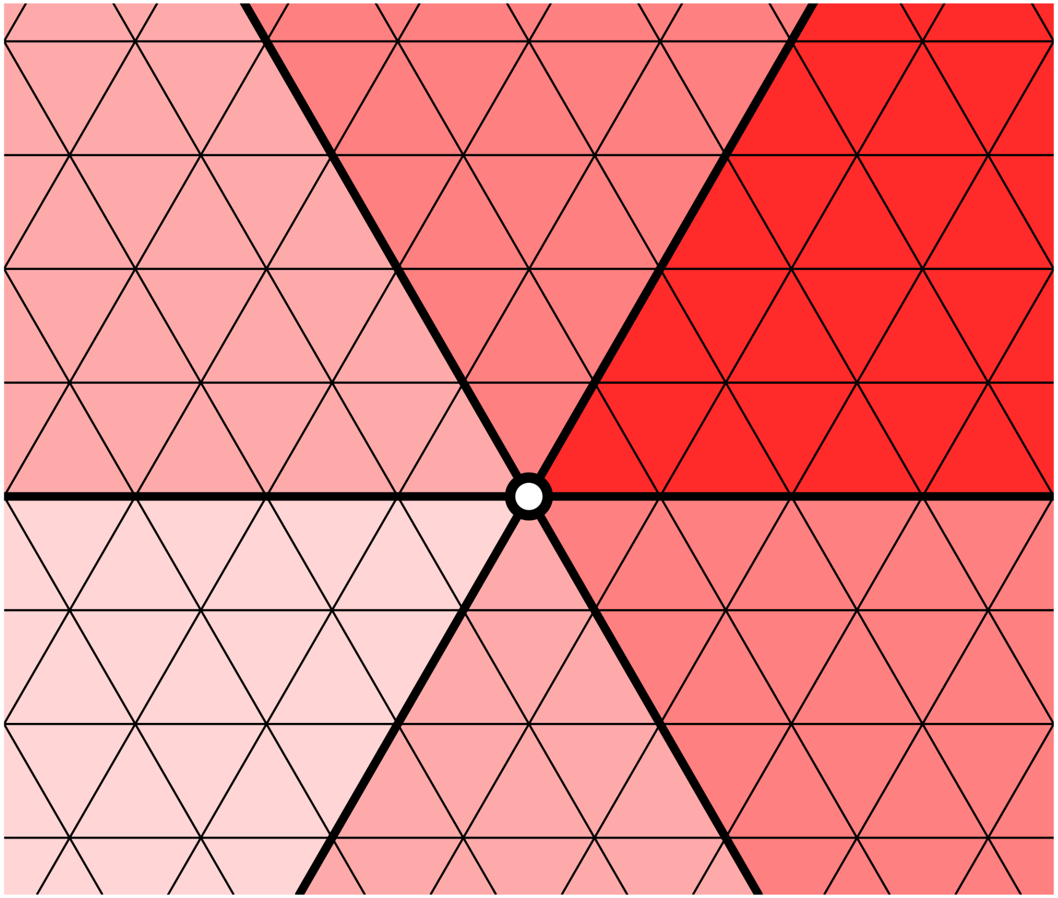
$=$ (Which cone are you in?) X (Where in the cone?)

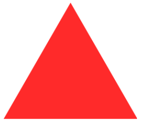
$=$ (permute the window notation) X (into increasing order)

zB

$$[6,-3,8,-1] = [3,1,4,2] \times [-3,-1,6,8]$$

Way 1 to think of $\tilde{\mathfrak{S}}_3$





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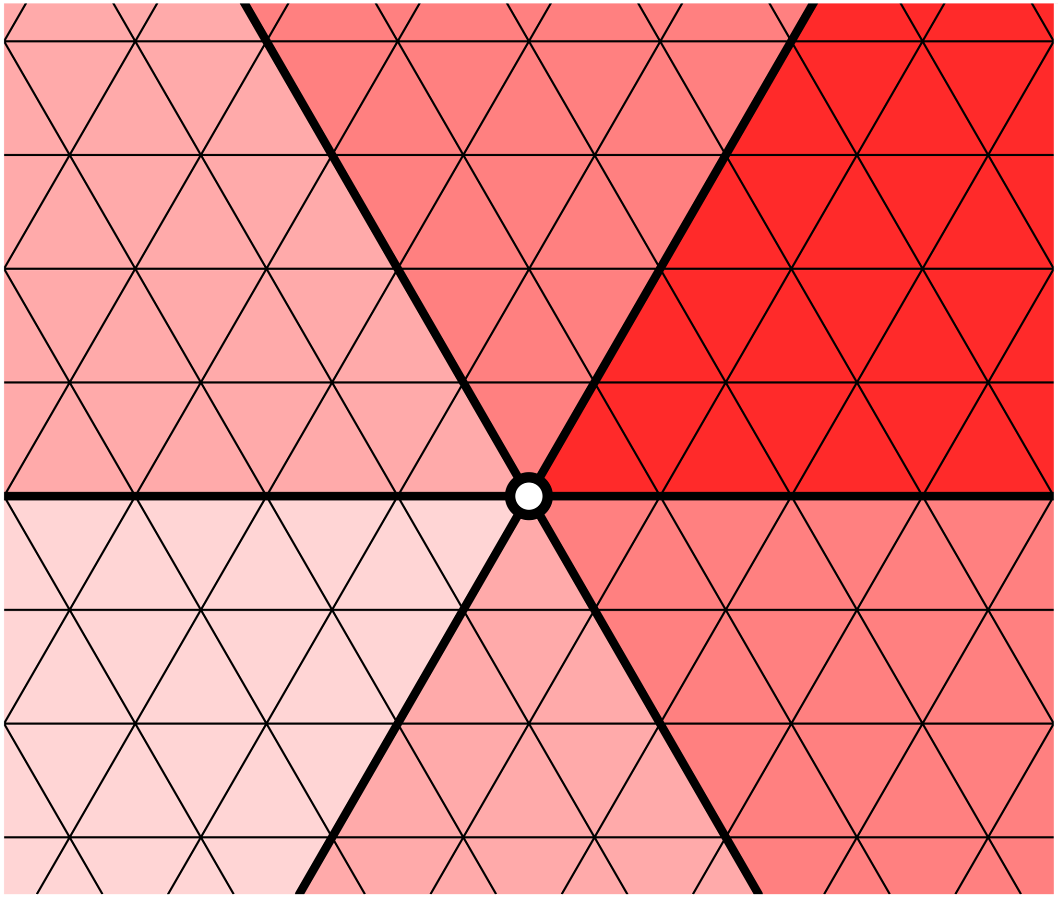
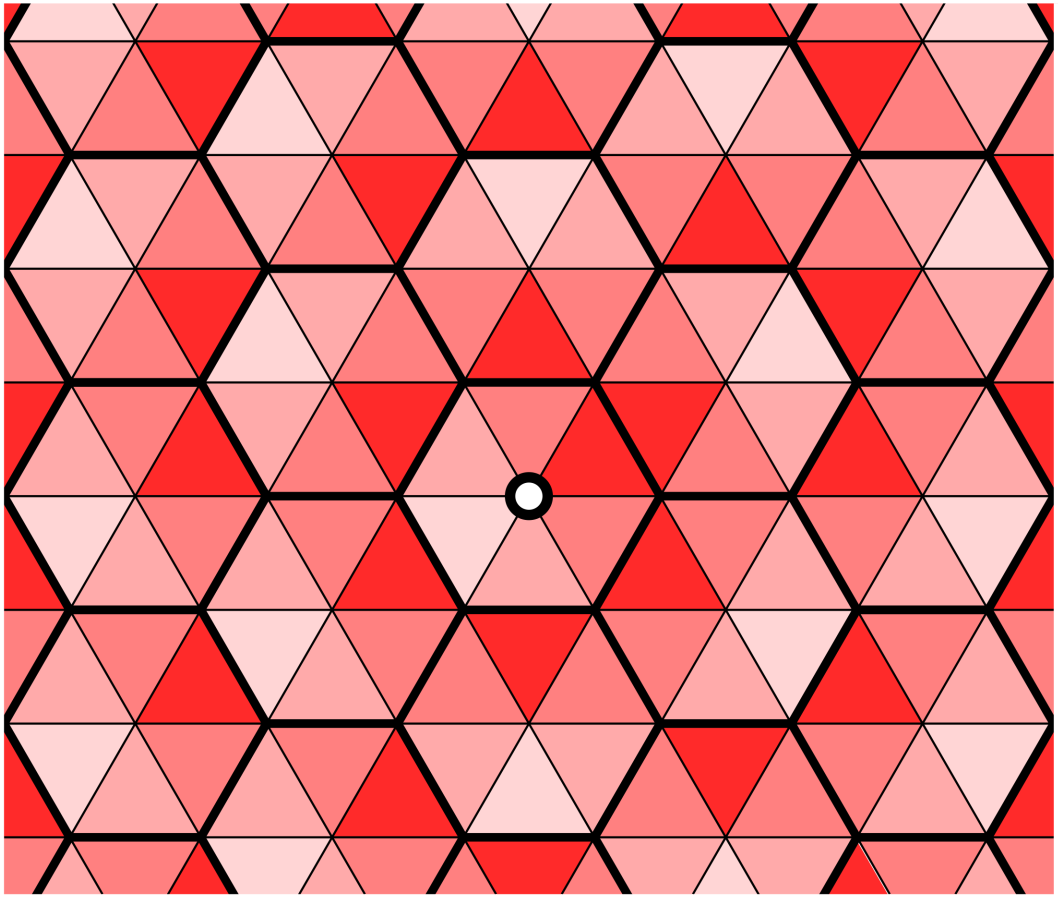
minimal coset rep

What happens if we invert everything?

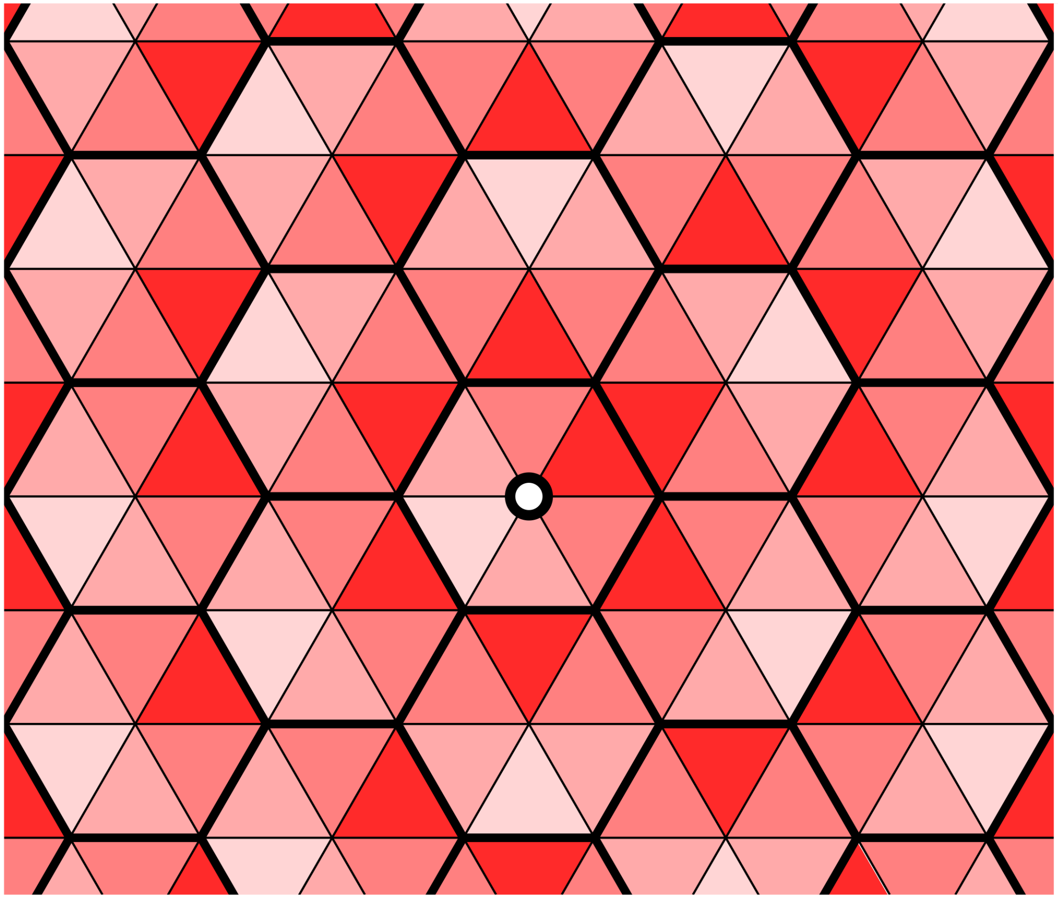


Invert!



Invert!

This is way 2 to think.



Invert! This is way 2 to think.

2. $\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes Q$

semi-direct product with the root lattice

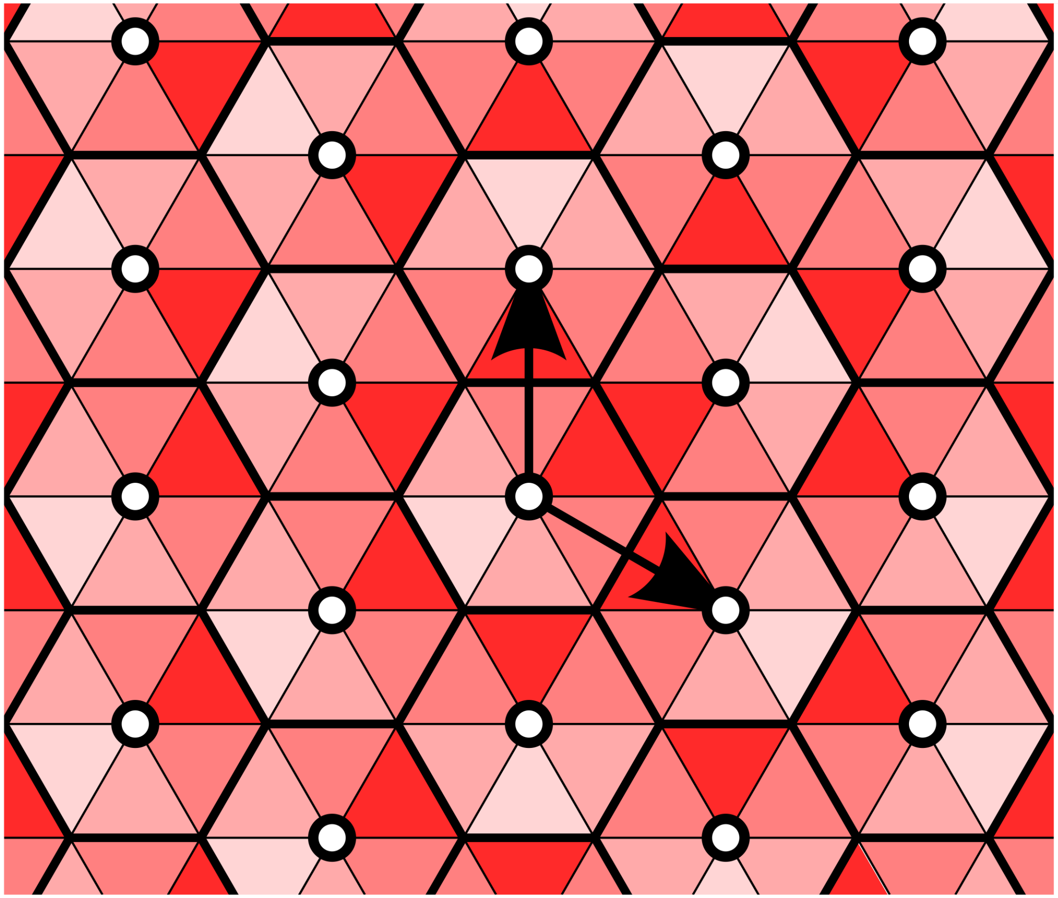
$$Q = \{(r_1, \ldots, r_n) \in \mathbb{Z}^n : \sum_i r_i = 0\}$$

In terms of window notation:

$$[6, -3, 8, -1] = (2, 1, 4, 3) + 4 \cdot (1, -1, 1, -1)$$

"finite permutation + *n* times a root"

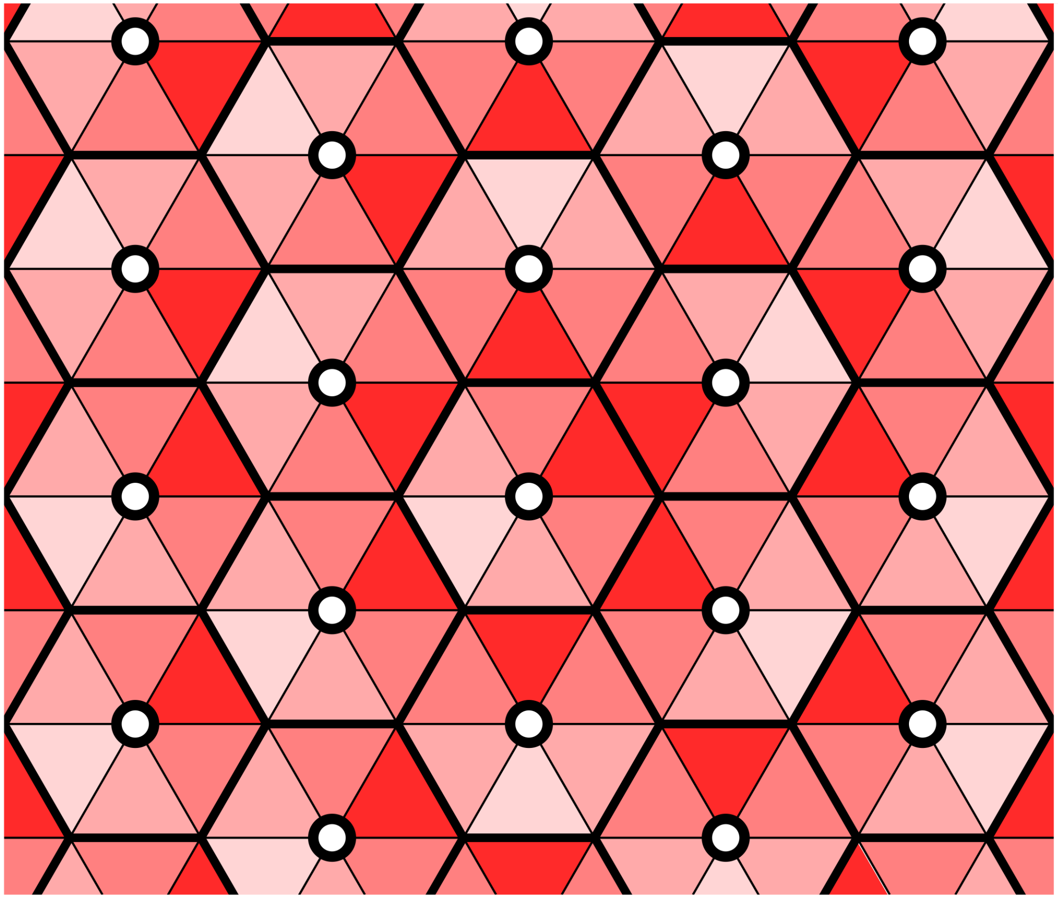
A root in each hexagon.



bijection:

$$Q \leftrightarrow \mathfrak{S}^n$$

Today: a NEW statistic on the root lattice.



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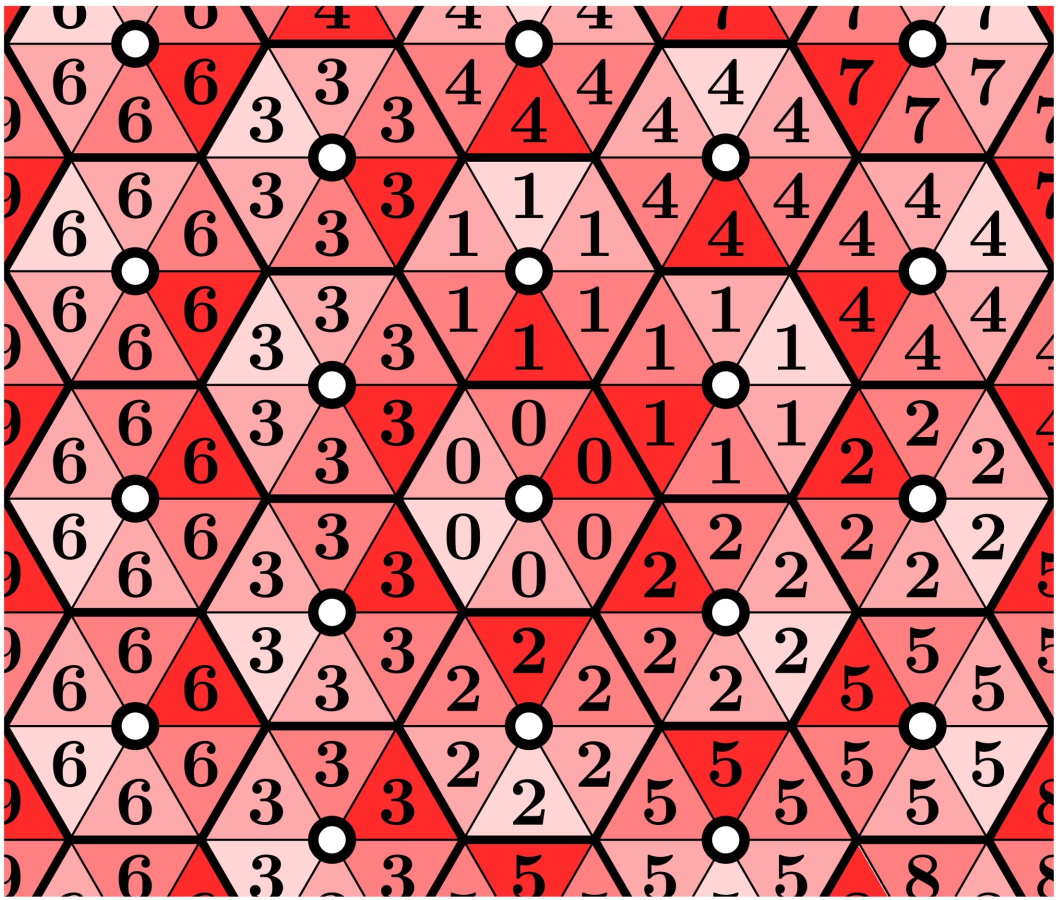


"It spirals!"

Call it "**ish**".

The definition:

Given $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in Q$

Let j be **maximal** such that r_j is **minimal**.

Then:

$$\mathsf{ish}(\mathbf{r}) := j - n(r_j + 1)$$

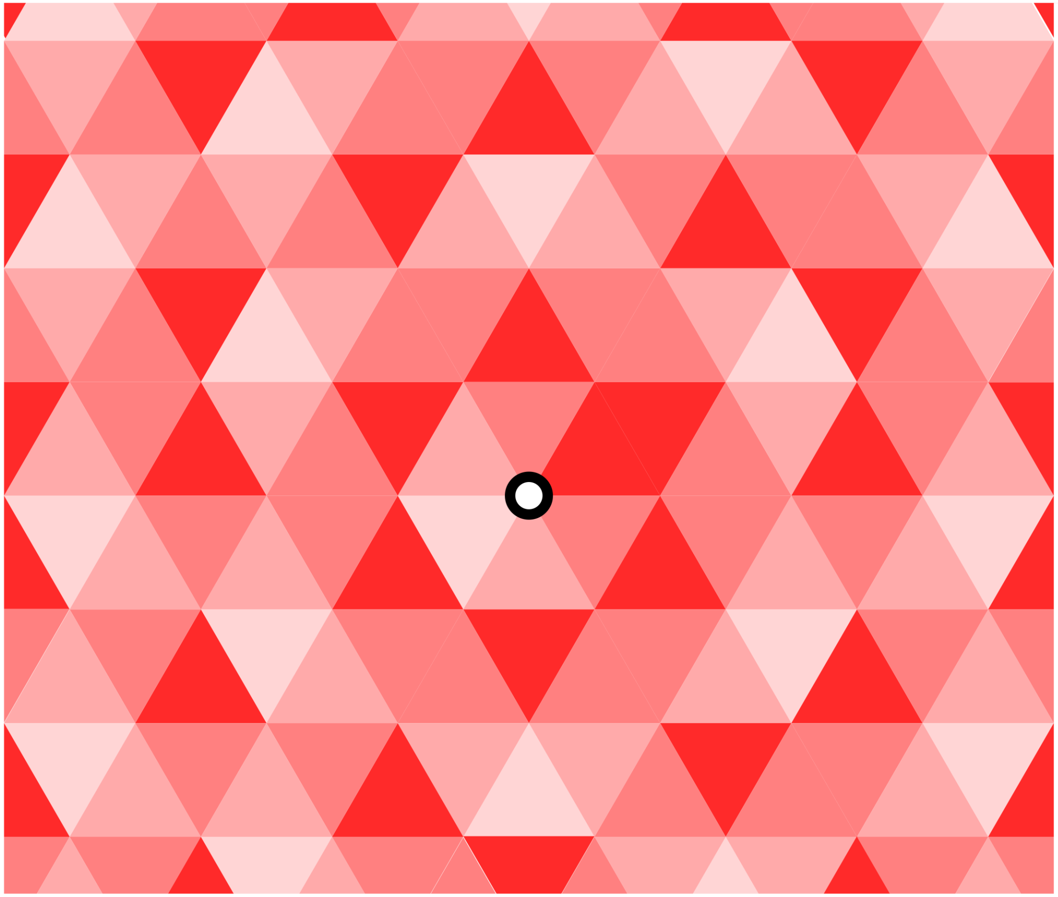
zB

$$\mathsf{ish}(2, -2, 2, -2, 0) = 4 - 5 \cdot (-2 + 1) = 9$$

$$n = 5$$
$$j = 4$$
$$r_j = -2$$

Part II: **Shi**

Start with a special simplex *D*.



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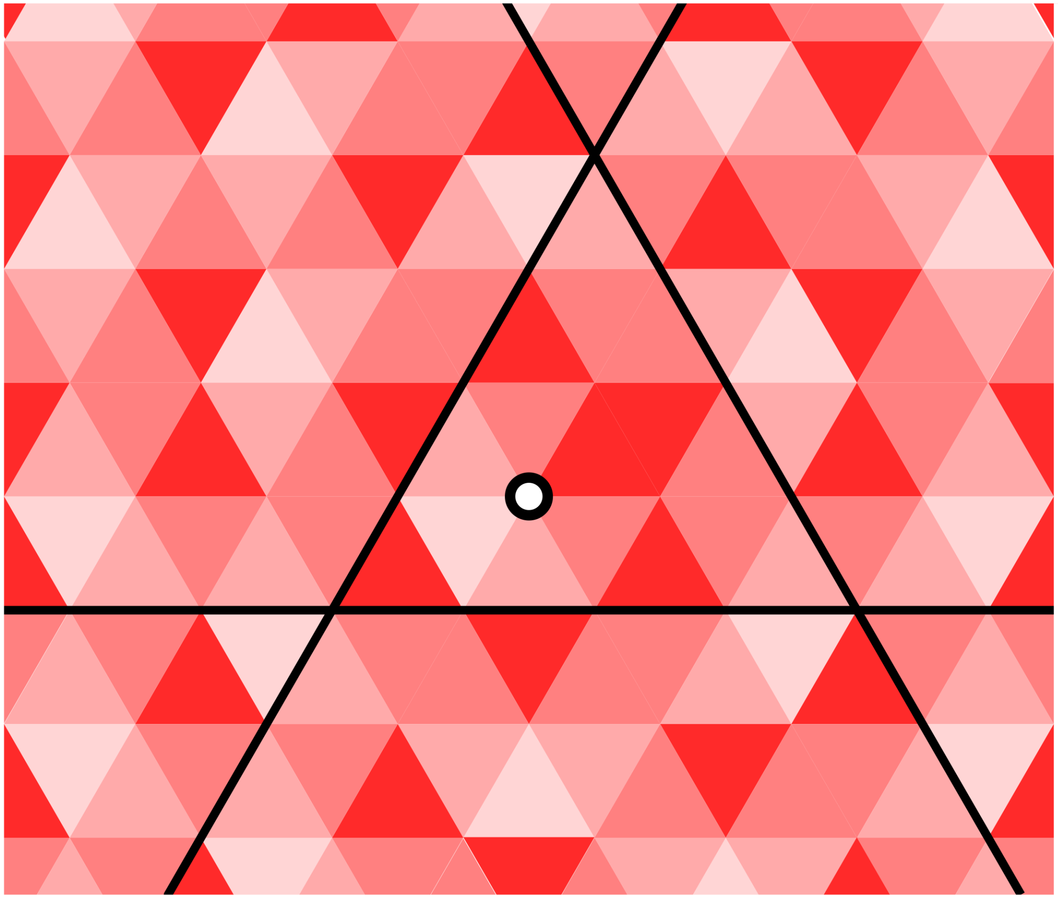

bounded by:

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_2 - x_3 &= -1 \\ &\vdots \\ x_{n-1} - x_n &= -1 \\ x_1 - x_n &= 2 \end{aligned}$$

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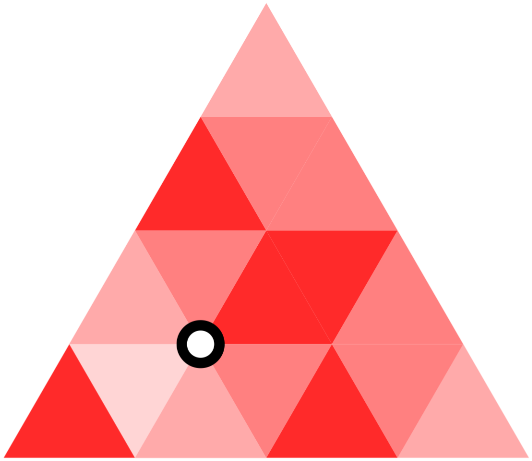
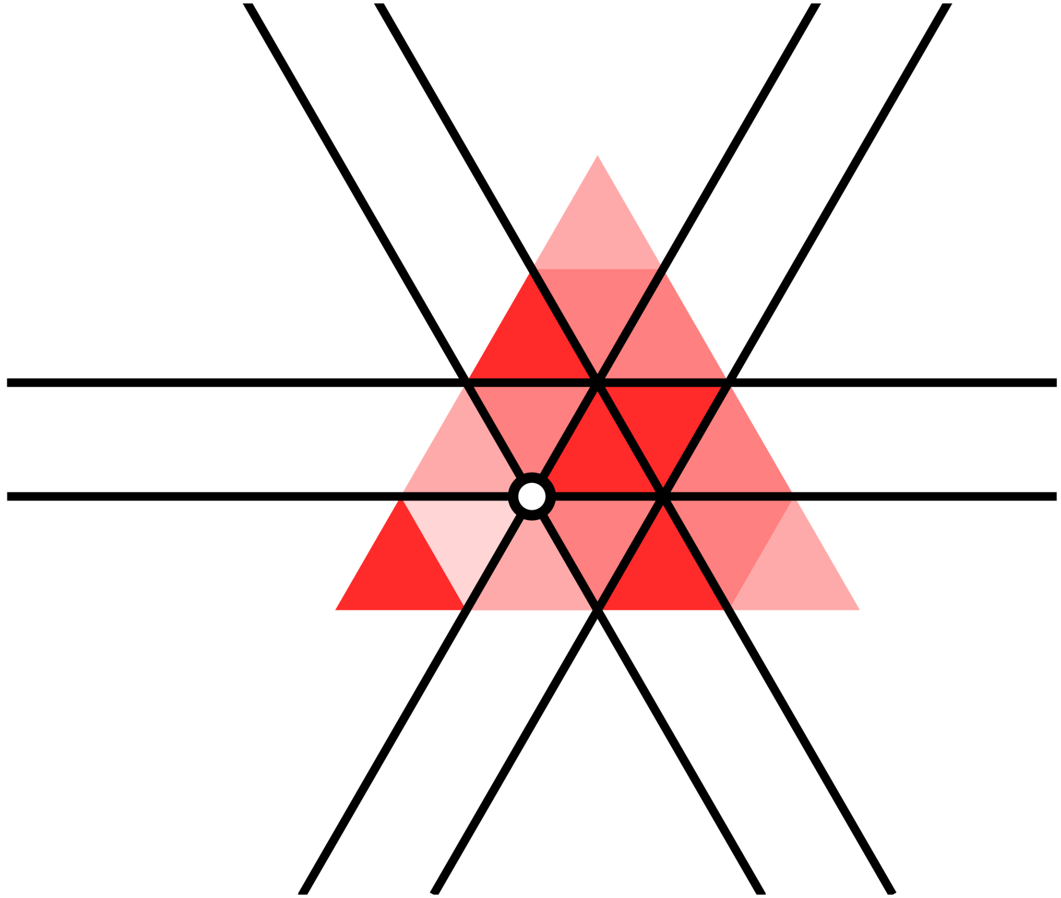


bounded by:

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_2 - x_3 &= -1 \\ &\vdots \\ x_{n-1} - x_n &= -1 \\ x_1 - x_n &= 2 \end{aligned}$$

Note: *D* contains $(n+1)^{n-1}$ alcoves.

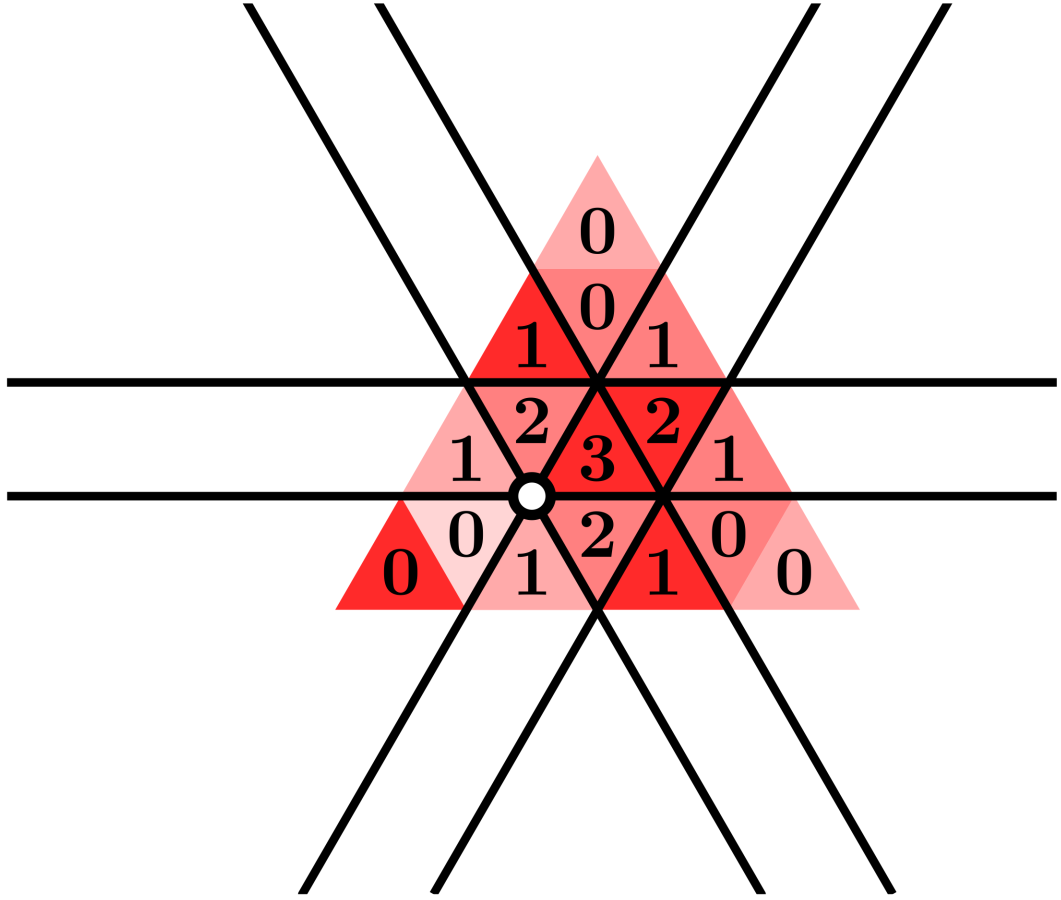
Next consider the “Shi hyperplanes”.



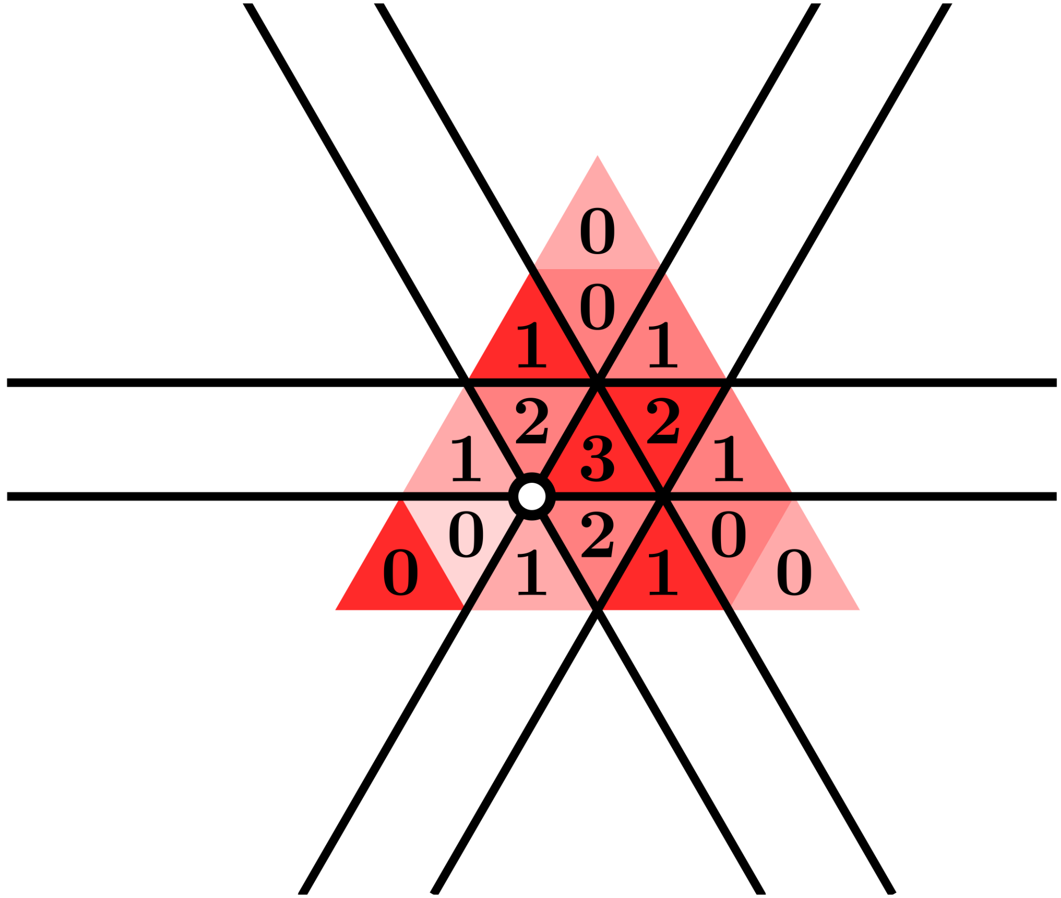
$$\mathsf{Shi}(n) :=$$

$$\begin{aligned} x_1 - x_2 &= 0, 1 \\ x_2 - x_3 &= 0, 1 \\ &\vdots \\ x_{n-1} - x_n &= 0, 1 \end{aligned}$$

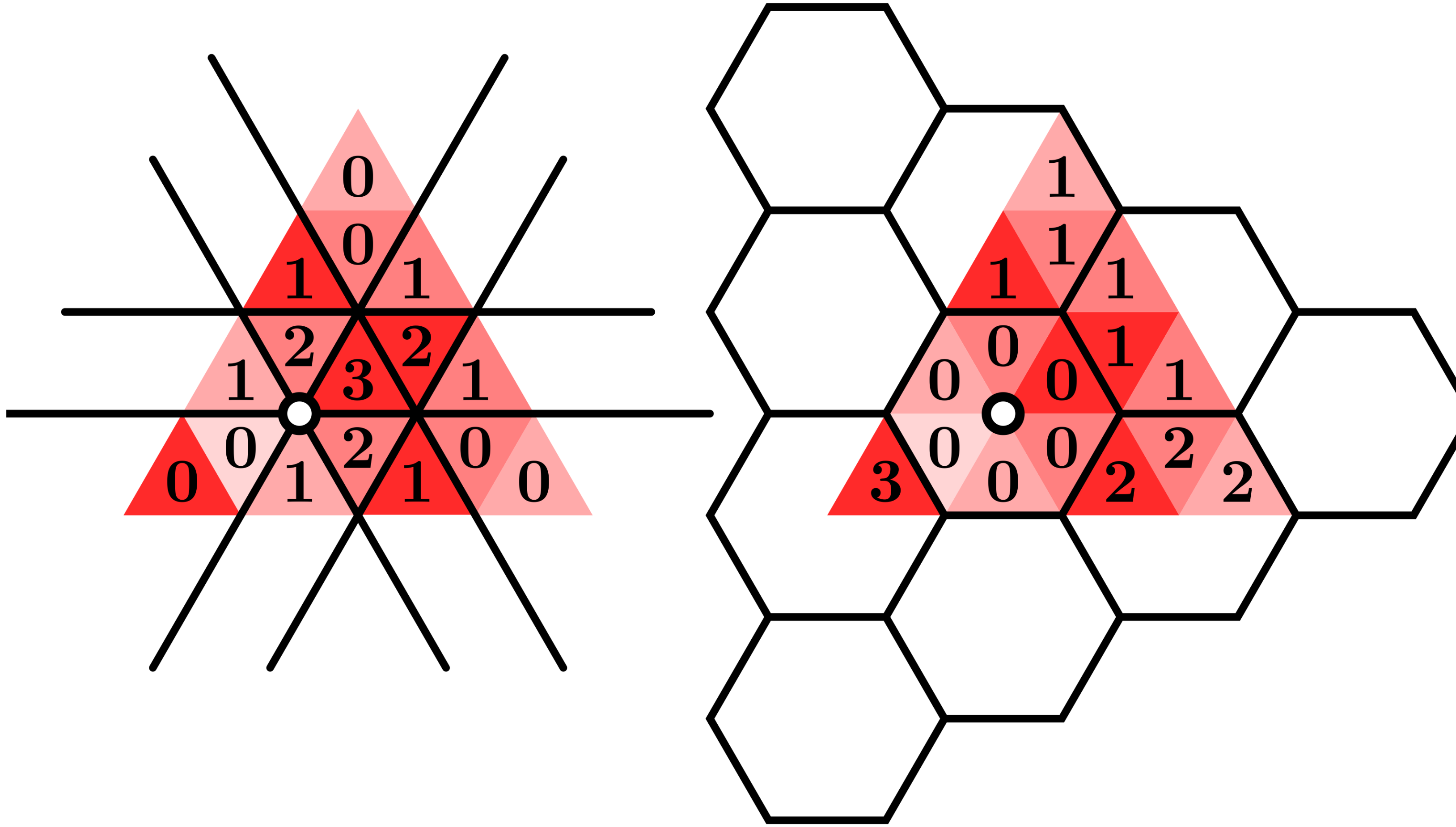
And their “distance enumerator”.



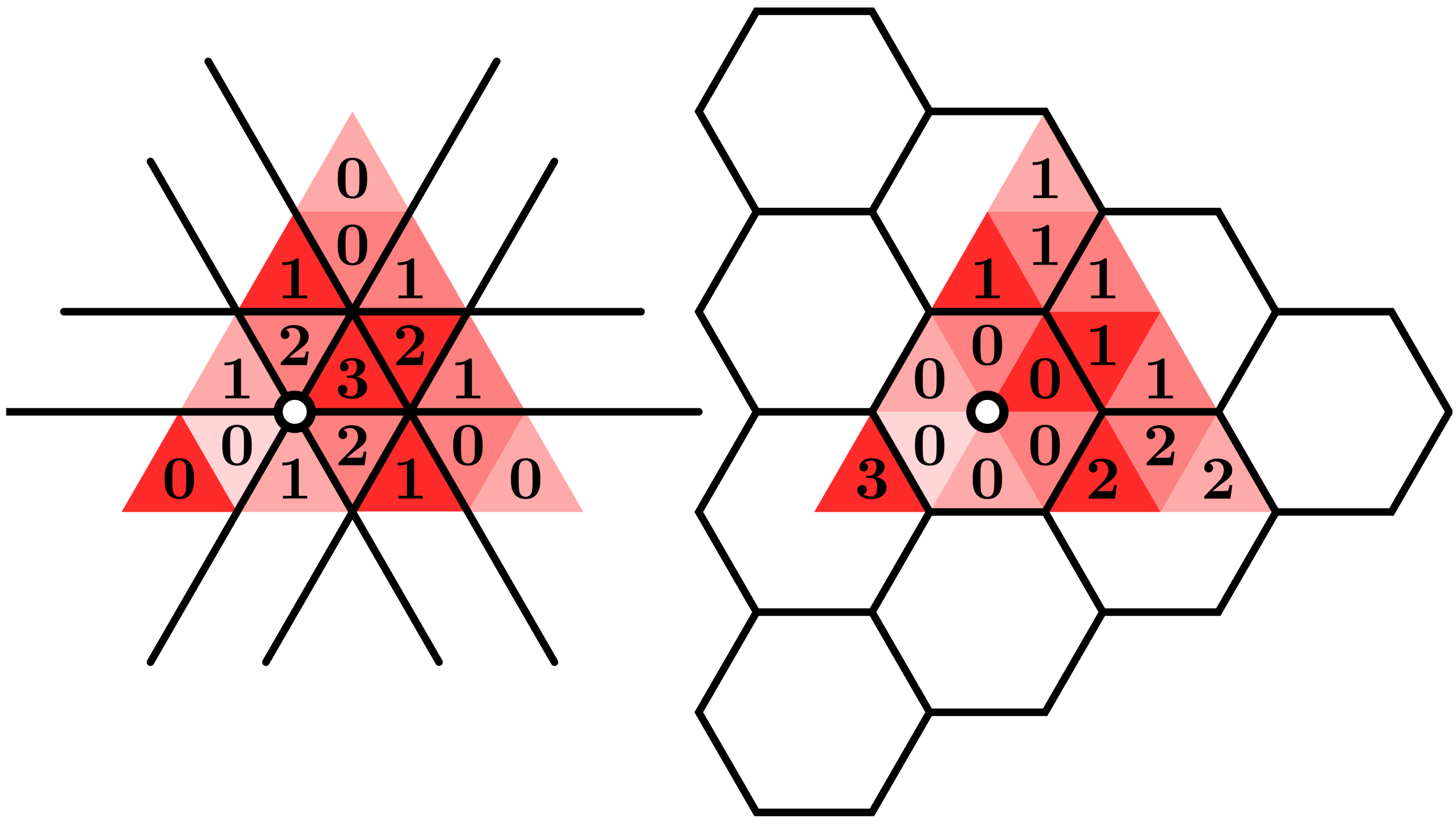
Call it “**shi**”.



Behold! **shi** and **ish** together.



Generating Function: $\mathsf{Shi}(n;q,t) = \sum q^{\mathsf{shi}} t^{\mathsf{ish}}$



Generating Function: $\mathsf{Shi}(n;q,t) = \sum q^{\mathsf{shi}} t^{\mathsf{ish}}$

zB

$$\mathsf{Shi}(3;q,t) = \begin{array}{c|cccc} q\backslash t & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 1 & \\ 2 & 2 & 1 & & \\ 3 & 1 & & & \end{array}$$

Conjectures:

- Symmetric in q and t.
- Equal to the Hilbert series of diagonal harmonics.

Theorem (me, 2009)

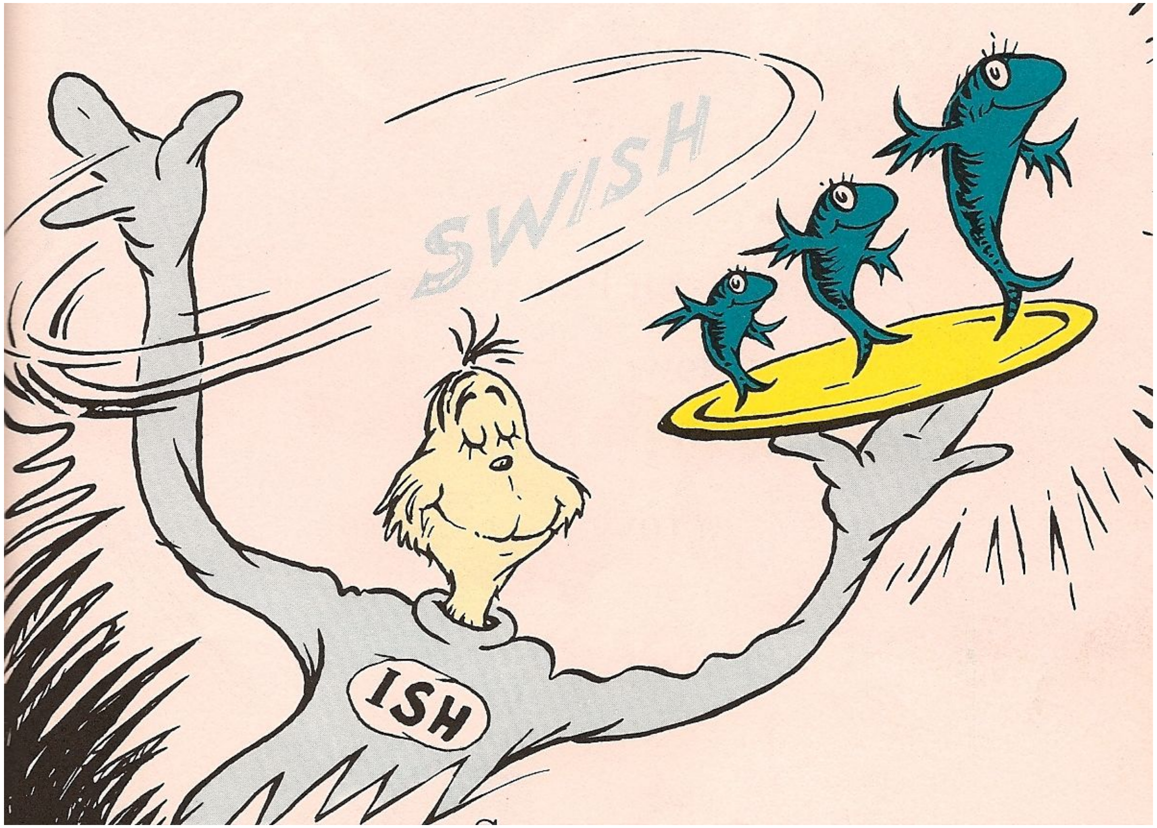
$\exists$ (at least) two natural maps to parking functions.

$$\begin{array}{ccccc} \mathrm{dinv} & & \mathrm{shi} & & \mathrm{area}' \\ & \longleftarrow & & \longrightarrow & \\ \mathrm{area} & & \mathrm{ish} & & \mathrm{bounce} \end{array}$$

"Haglund-Haiman-Loehr statistics"

$\Rightarrow$ Various fun corollaries!

Part III: **Nabla**



A **family** of simplices.

Given: p coprime to n with quotient and remainder.

$$p = qn + r$$

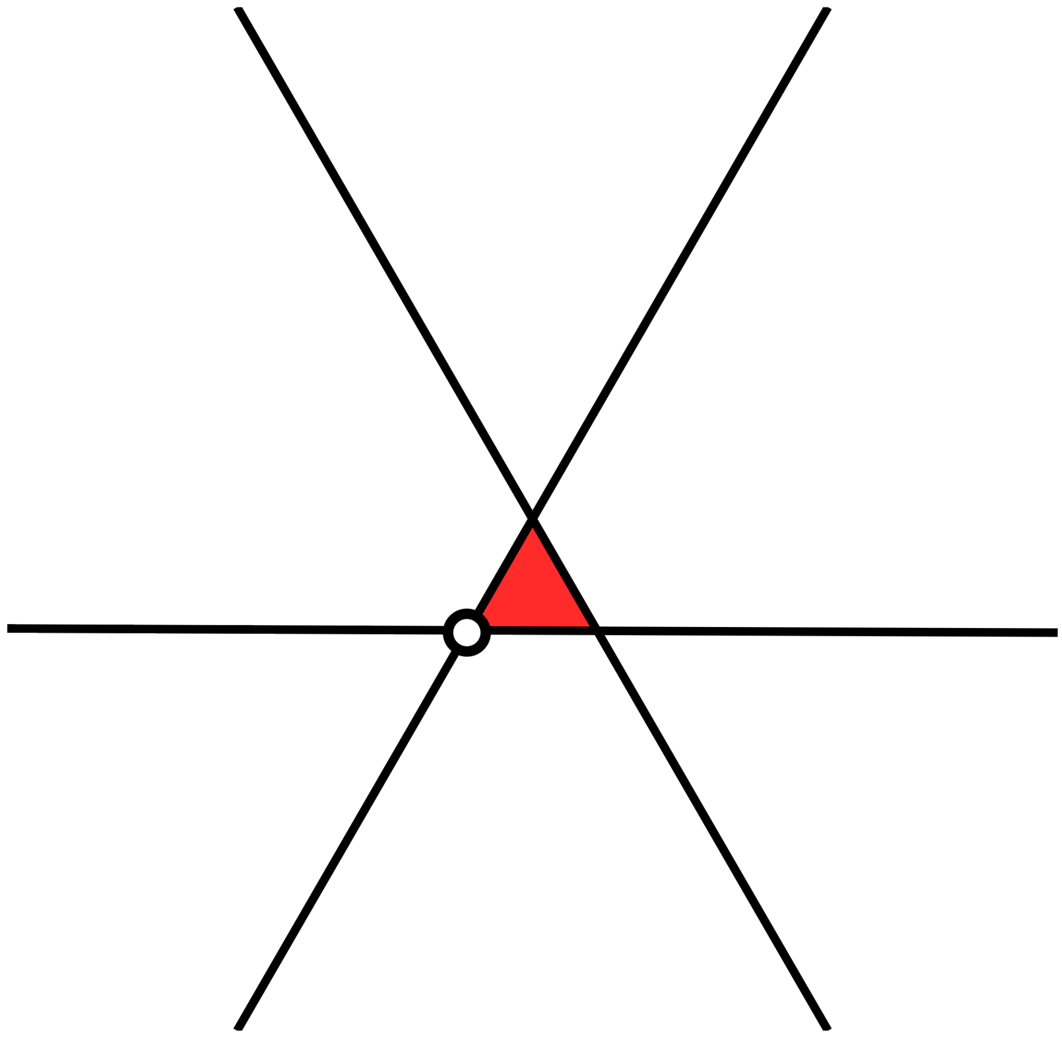
Let $D^p(n)$ be the simplex bounded by

$$\{x_i - x_j = q : i - j = r\}$$

$$\bigcup \{x_i - x_j = q + 1 : i - j = r - n\}$$

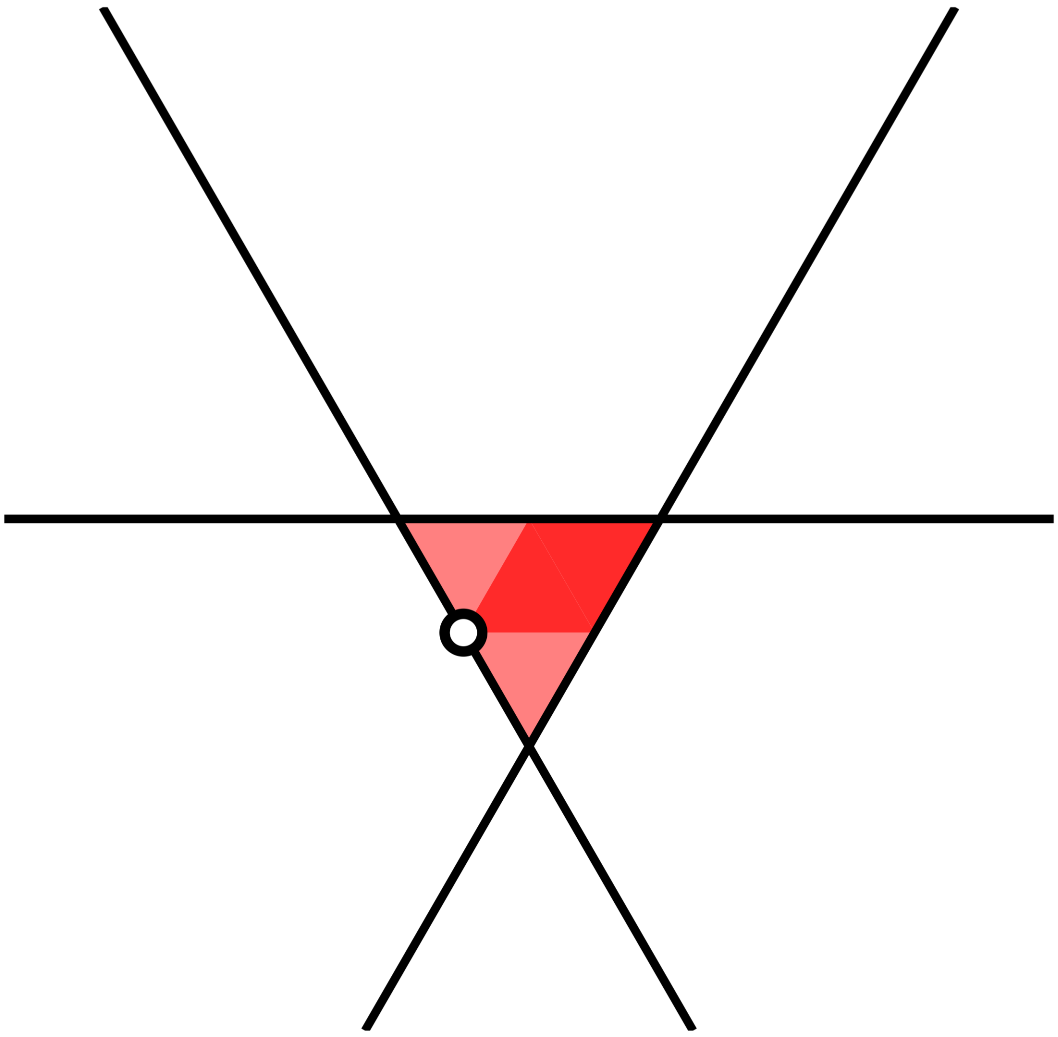
zB: $D = D^{n+1}(n)$

Observe: $D^1(3)$



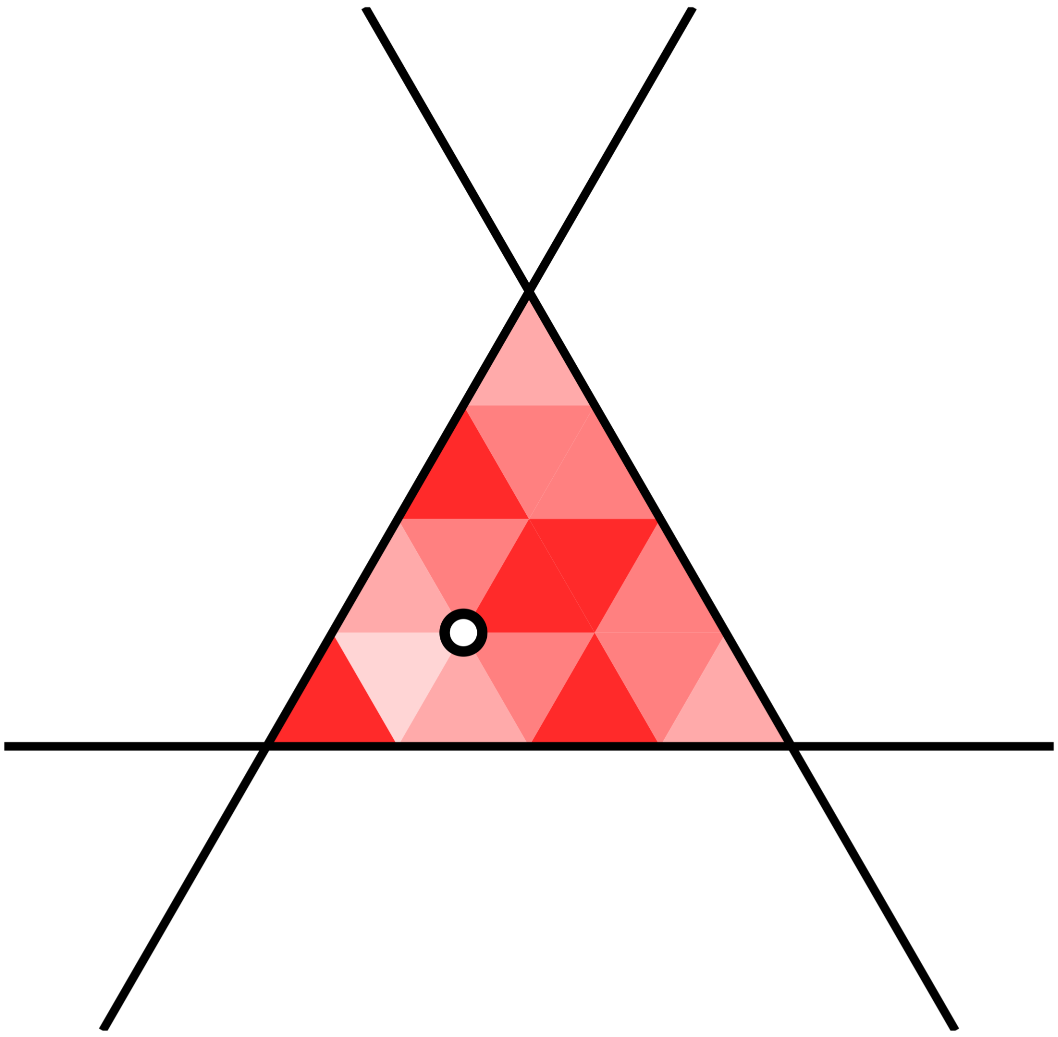
1 alcove

Observe: $D^2(3)$



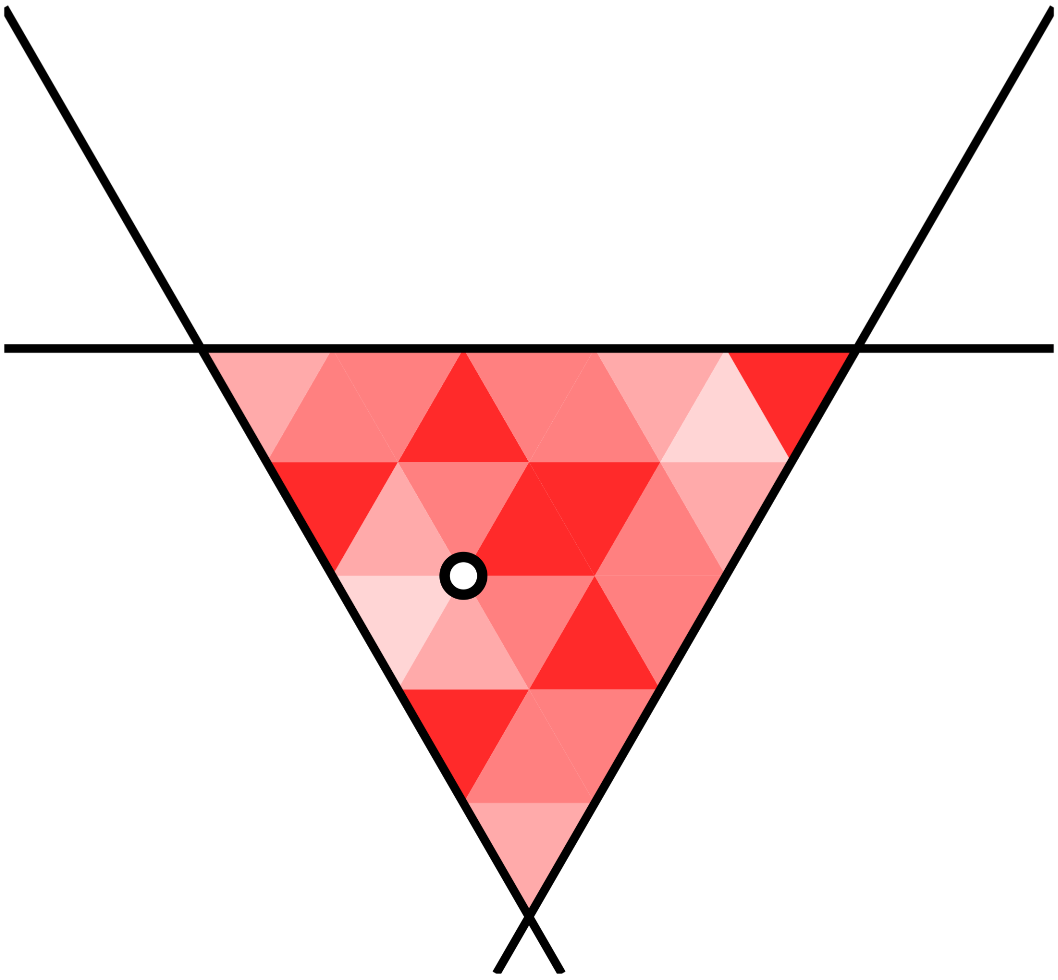
4 alcoves

Observe: $D^4(3)$



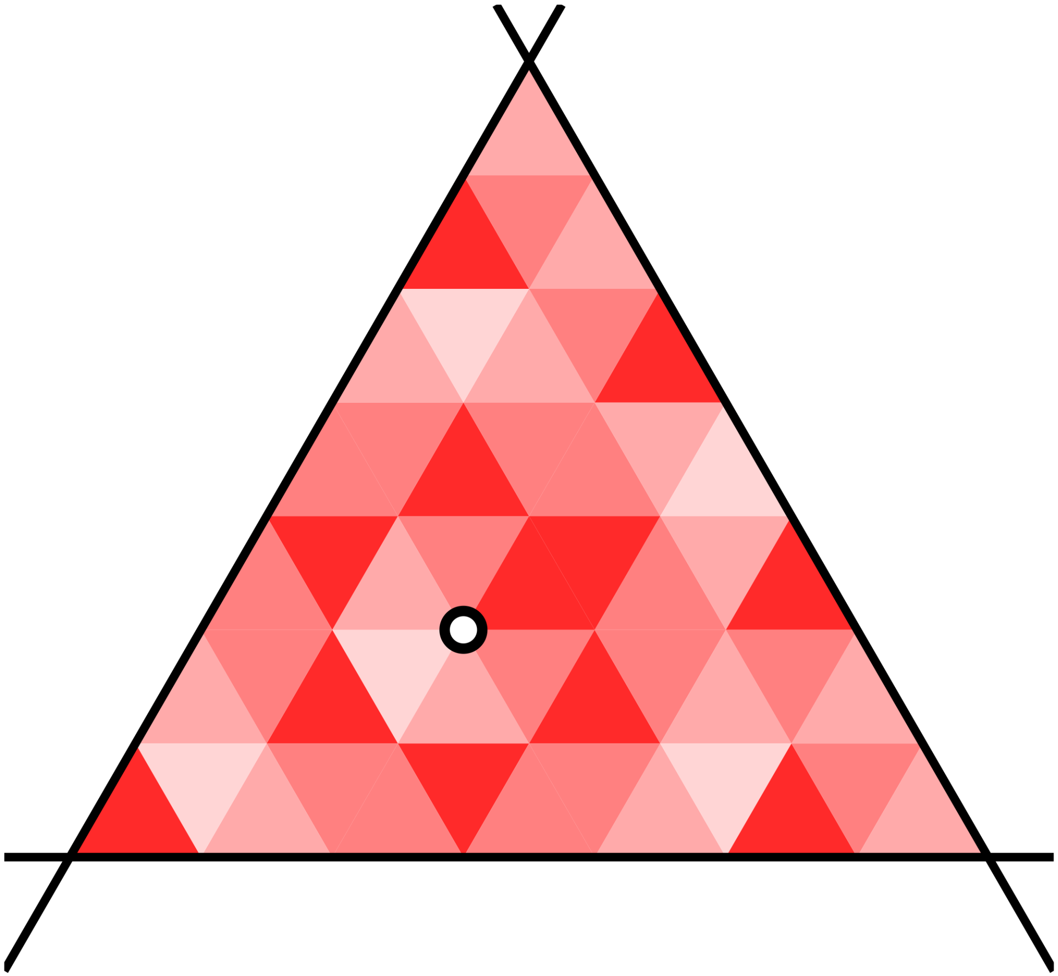
16 alcoves

Observe: $D^5(3)$



25 alcoves

Observe: $D^7(3)$



49 alcoves

Theorem (Sommers, 2005)

$$D^p(n) \approx pA_\circ$$

“a dilation of the fundamental alcove”

Hence $D^p(n)$ contains p^{n-1} alcoves.

“parking functions?”

Q: Can I extend **shi** and **ish** to $D^p(n)$?

A: Well..... yes, when $p = mn \pm 1$

Do you like ∇ ?

∇ = Bergeron-Garsia nabla operator

e_n = elementary symmetric function

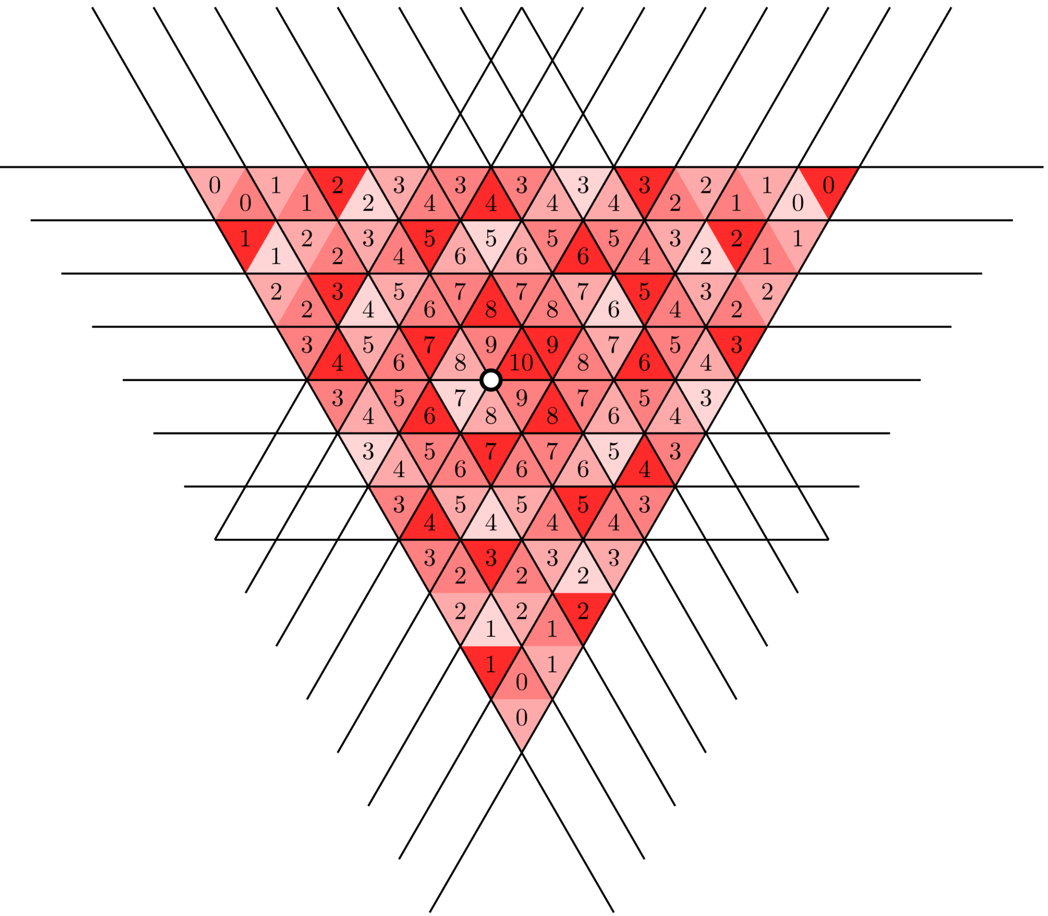
∇e_n = Frobenius series of diagonal harmonics

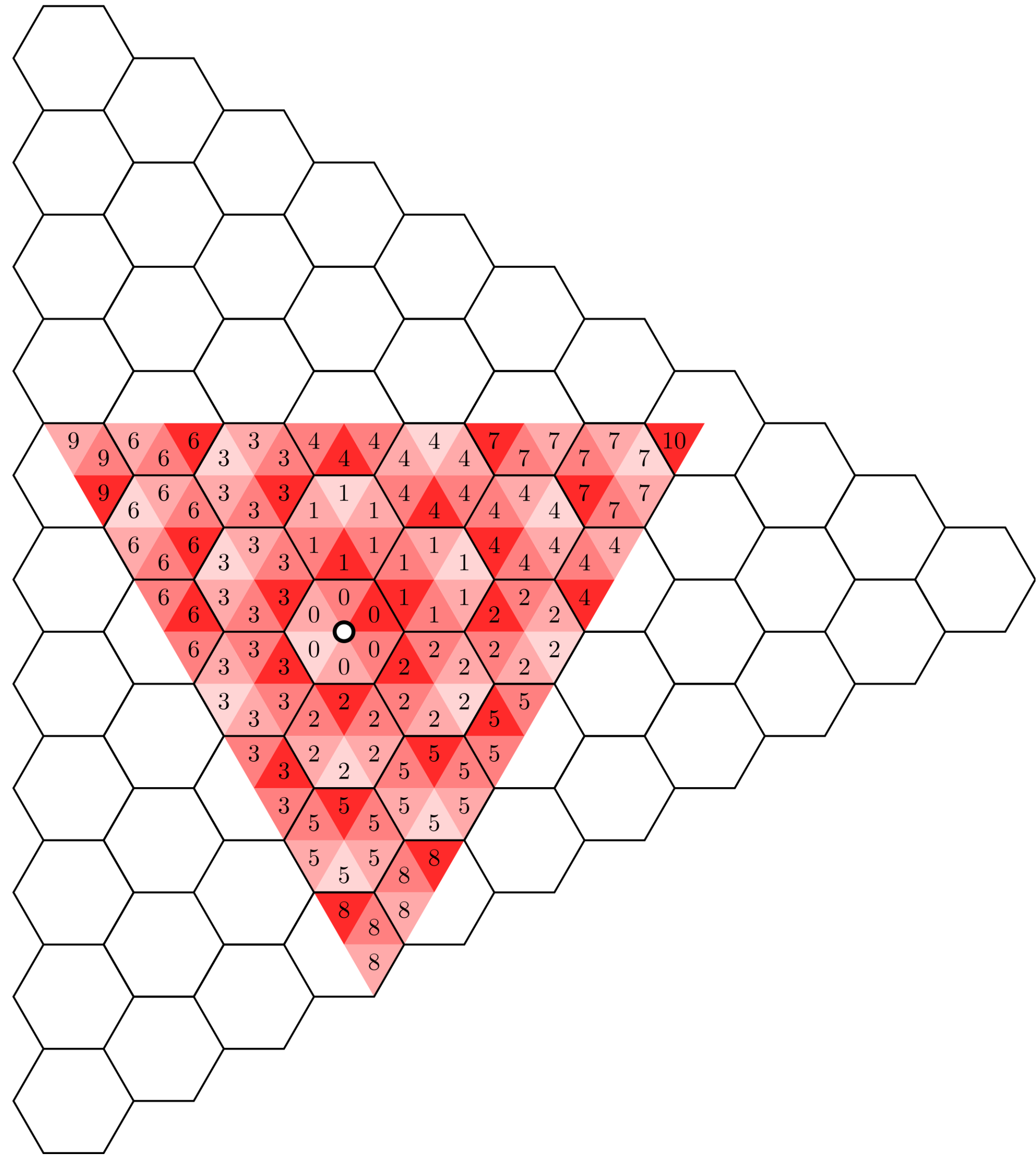
Conjecture (me, 2010): Given m>0, we have

$$\sum_{D^{mn+1}} q^{\mathsf{shi}} t^{\mathsf{ish}} \approx \nabla^m e_n$$

$$\sum_{D^{mn-1}} q^{\mathsf{shi}} t^{\mathsf{ish}} \approx \nabla^{-m} e_n$$

Example: $\nabla^{-4} e_3$





Questions:

- Other root systems?
- The full Frobenius series?
- Other coprime numbers p?

Fin.