Rational Associahedra

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York University
Applied Algebra Seminar
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What is a Catalan Number?

Convention

Given $x \in \mathbb{Q} \setminus [-1, 0]$ there exist unique positive coprime $a, b \in \mathbb{Z}$ with

$$x = \frac{a}{b-a}.$$ 

We will always identify $x \leftrightarrow (a, b)$.

Examples: Given $n \geq 1$ we have
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**Examples:** Given \( n \geq 1 \) we have

\[
x = -n = \frac{n}{-1} = \frac{n}{(n - 1) - n} \leftrightarrow (n, n - 1) \quad \text{need } n \geq 2
\]
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Examples: Given \( n \geq 1 \) we have

\[
x = -\frac{1}{n} = \frac{1}{-n} = \frac{1}{(-n + 1) - 1} \leftrightarrow \text{impossible!}
\]
What is a Catalan Number?

Definition

For each \( x \in \mathbb{Q} \setminus [-1, 0] \) we define the **Catalan number**:

\[
\text{Cat}(x) = \text{Cat}(a, b) := \frac{1}{a + b} \binom{a + b}{a, b} = \frac{(a + b - 1)!}{a!b!}.
\]

Claim: This is an integer. (Proof postponed.)

Example:

\[
\text{Cat} \left( \frac{5}{3} \right) = \text{Cat} \left( \frac{5}{8 - 5} \right) = \text{Cat}(5, 8) = \frac{12!}{5!8!} = 99.
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- **Eugène Charles Catalan (1814-1894)**
  
  $(a, b) = (n, n + 1)$ gives the good old **Catalan number**:
  
  $$
  \text{Cat}(n) = \text{Cat} \left( \frac{n}{(n + 1) - n} \right) = \frac{1}{2n + 1} \binom{2n + 1}{n}.
  $$

- **Nicolaus Fuss (1755-1826)**
  
  $(a, b) = (n, kn + 1)$ gives the **Fuss-Catalan number**:
  
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  \text{Cat} \left( \frac{n}{(kn + 1) - n} \right) = \frac{1}{(k + 1)n + 1} \binom{(k + 1)n + 1}{n}.
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By definition we have $\text{Cat}(a, b) = \text{Cat}(b, a)$, which implies that

$$\text{Cat}(x) = \text{Cat}(a, b) = \text{Cat}(b, a) = \text{Cat}(-x - 1).$$

This implies that for $0 < x \in \mathbb{Q}$ (i.e. $a < b$) we have

$$\text{Cat} \left( \frac{1}{x - 1} \right) = \text{Cat} \left( \frac{x}{1 - x} \right).$$

We will call this the derived Catalan number:

$$\text{Cat}'(x) := \text{Cat} \left( \frac{1}{x - 1} \right) = \text{Cat} \left( \frac{x}{1 - x} \right).$$
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Note that $x > 0 \iff \frac{1}{x} > 0$ and we have

$$\text{Cat}'\left(\frac{1}{x}\right) = \text{Cat} \left(\frac{1}{(1/x) - 1}\right) = \text{Cat} \left(\frac{x}{1 - x}\right) = \text{Cat}'(x).$$

We call this rational duality:

$$\text{Cat}'(x) = \text{Cat}'\left(\frac{1}{x}\right).$$

In terms of coprime $0 < a < b$ this translates to

$$\text{Cat}'(a, b) = \text{Cat}'(b - a, b).$$

(This will appear later as Alexander duality of rational associahedra.)
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Euclidean Algorithm

Observation

Given $0 < a < b$ coprime, we observe that

\[
\text{Cat}'(a, b) = \frac{1}{b} \binom{b}{a} = \begin{cases} 
\text{Cat}(a, b - a) & \text{for } a < (b - a) \\
\text{Cat}(b - a, a) & \text{for } (b - a) < a
\end{cases}
\]

This allows us to define a sequence

\[
\text{Cat}(x) \mapsto \text{Cat}'(x) \mapsto \text{Cat}''(x) \mapsto \cdots
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which is a “Categorification” of the Euclidean algorithm.
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Euclidean Algorithm

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

\[
\begin{align*}
\text{Cat}(5, 8) &= 99,
\text{Cat'}(5, 8) &= \text{Cat}(3, 5) = 7,
\text{Cat''}(5, 8) &= \text{Cat'}(3, 5) = \text{Cat}(2, 3) = 2,
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How to put it in Sloane’s OEIS

Suggestion

The Calkin-Wilf sequence is defined by $q_1 = 1$ and

$$ q_n := \frac{1}{2 \lfloor q_{n-1} \rfloor - q_{n-1} + 1}. $$

Theorem: $(q_1, q_2, \ldots) = \mathbb{Q}_{>0}$.

Proof: See “Proofs from THE BOOK”, Chapter 17.

Study the function $n \mapsto \text{Cat}(q_n)$.

<table>
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<tr>
<th>$q$</th>
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Well, that was fun.
The Prototype: Rational Dyck Paths
Consider the “Dyck paths” in an $a \times b$ rectangle.

Example $(a, b) = (5, 8)$
Again let $0 < x = a/(b - a)$ with $0 < a < b$ coprime.

Example $(a, b) = (5, 8)$
Let $\mathcal{D}(x) = \mathcal{D}(a, b)$ denote the set of Dyck paths.

Example $(a, b) = (5, 8)$
Theorem (Grossman 1950, Bizley 1954)

For \( a, b \) coprime, the number of Dyck paths is the Catalan number:

\[
|D(x)| = \text{Cat}(x) = \frac{1}{a + b} \binom{a + b}{a, b}.
\]

- Claimed by Grossman (1950), “Fun with lattice points, part 22”.
- Proof: Break \( \binom{a+b}{a,b} \) lattice paths into cyclic orbits of size \( a + b \). Each orbit contains a unique Dyck path.
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The number of Dyck paths with \( k \) vertical runs equals

\[
\text{Nar}(x; k) := \frac{1}{a} \binom{a}{k} \binom{b - 1}{k - 1}.
\]

Call these the \textbf{Narayana numbers}.

And the number with \( r_j \) vertical runs of length \( j \) equals

\[
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Let $n \geq 0$ and consider a convex $(n+2)$-gon $C$. Let $\text{Ass}(n)$ be the abstract simplicial complex with

- vertices = chords of $C$
- faces = noncrossing sets of chords of $C$
- max. faces = triangulations of $C$

Theorem (Milnor, Haiman, C. Lee, etc.)

$\text{Ass}(n)$ is a polytope.
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- vertices = chords of \( C \)
- faces = noncrossing sets of chords of \( C \)
- max. faces = triangulations of \( C \)

Theorem (Milnor, Haiman, C. Lee, etc.)
\( \text{Ass}(n) \) is a polytope.
The Classical Associahedron

Definition

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Theorem (Milnor, Haiman, C. Lee, etc.)

$Ass(n)$ is a polytope.
Example: Here is Ass(4).
The Classical Associahedron

**Theorem (Euler, 1751)**

*The f-vector and h-vector of* \( \text{Ass}(n) \) *are given by the Kirkman numbers*

\[
\text{Kirk}(n; k) = \frac{1}{n} \binom{n}{k} \binom{n+k}{k-1}
\]

*and the Narayana numbers*

\[
\text{Nar}(n; k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.
\]
Example: Here are the *f*-vector and *h*-vector of Ass(4).

```
1
1  6
1  7  6
1  8 13  1
1  9 21 14
```
The Rational Associahedron?

Question

Given $0 < x = a/(b - a)$ with $0 < a < b$ coprime, can one define a “rational associahedron”

$$\text{Ass}(x) = \text{Ass}(a, b)$$

with the “correct” numerology and structure?

Answer

Yes.
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The Rational Associahedron?

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with the “correct” numerology and structure?

Answer

Yes.
To define a “rational triangulation” . . .

- Start with a Dyck path. Here \((a, b) = (5, 8)\).
To define a “rational triangulation” . . .

- Label the columns by \( \{1, 2, \ldots, b + 1\} \).
To define a “rational triangulation” . . .

- Shoot lasers from the bottom left with slope $a/b$. 
To define a “rational triangulation” . . .

- Lift the lasers up.
To define a “rational triangulation” ...
To define a “rational triangulation” . . .

- We have constructed \( \text{Cat}(a, b) \) many “rational triangulations” of a convex \((b + 1)\)-gon, and each of them has \( a - 1 \) chords.
The Rational Associahedron

**Definition**

Given $0 < x = a/(b - a)$, let $\text{Ass}(x) = \text{Ass}(a, b)$ be the abstract simplicial complex whose maximal faces are the “rational triangulations”.

**Geometric Realization**

Note that $\text{Ass}(a, b)$ is a pure $(a - 1)$-dimensional subcomplex of the $(b - 1)$-dimensional polytope $\text{Ass}(b - 1)$.
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The Rational Associahedron

Theorems (with B. Rhoades and N. Williams)

- Ass\((n, n + 1)\) is the **classical associahedron** \(\text{Ass}(n)\).
- Ass\((n, (k - 1)n + 1)\) is the **generalized cluster complex** of Athanasiadis-Tzanaki and Fomin-Reading.
- Ass\((x)\) has \(\text{Cat}(x)\) max. faces and **Euler characteristic** \(\text{Cat}'(x)\).
- Ass\((x)\) is **shellable** and hence homotopy equivalent to a wedge of \(\text{Cat}'(x)\) many \((a - 1)\)-dimensional spheres.
- Ass\((x)\) has **\(h\)-vector** \(\text{Nar}(x; k) = \frac{1}{a} \binom{a}{k} \binom{b-1}{k-1}\).
- Hence its **\(f\)-vector** is given by the **rational Kirkman numbers**:

\[
\text{Kirk}(x; k) := \frac{1}{a} \binom{a}{k} \binom{b + k - 1}{k - 1}.
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- Ass$(x)$ has **$h$-vector** Nar$(x; k) = \begin{pmatrix} a \\ k \end{pmatrix} \begin{pmatrix} b - 1 \\ k - 1 \end{pmatrix}$.
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  $$Kirk(x; k) := \frac{1}{a} \begin{pmatrix} a \\ k \end{pmatrix} \begin{pmatrix} b + k - 1 \\ k - 1 \end{pmatrix}.$$
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- \text{Ass}(x)\) has \textbf{h-vector} \text{Nar}(x; \, k) = \frac{1}{a} \binom{a}{k} \binom{b - 1}{k - 1}.
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\[ \text{Kirk}(x; \, k) := \frac{1}{a} \binom{a}{k} \binom{b + k - 1}{k - 1}. \]
Observation

Note that $\text{Ass}(b - 1)$ has this many vertices:

$$\binom{b + 1}{2} - (b + 1) = \frac{(b + 1)b}{2} - \frac{2(b + 1)}{2} = \frac{(b - 2)(b + 1)}{2}.$$ 

For all $0 < a < b$ coprime, the subcomplexes $\text{Ass}(a, b)$ and $\text{Ass}(b - a, b)$ bipartition the vertices of $\text{Ass}(b - 1)$ because

$$\frac{(a - 1)(b + 1)}{2} + \frac{(b - a - 1)(b + 1)}{2} = \frac{(b - 2)(b + 1)}{2}.$$
Example: Here are subcomplexes $\text{Ass}(2, 5)$ and $\text{Ass}(3, 5)$ in $\text{Ass}(4)$. 
Conjecture (with B. Rhoades and N. Williams)

We know that $\text{Ass}(a, b)$ and $\text{Ass}(b - a, b)$ have the same number of homotopy spheres (of complementary dimensions) because

$$\text{Cat}'(a, b) = \text{Cat}'(b - a, b).$$

We conjecture that the homotopy spheres are “intertwined” in a nice way. In particular, we conjecture that $\text{Ass}(a, b)$ and $\text{Ass}(b - a, b)$ are Alexander dual inside the sphere $\text{Ass}(b - 1)$.

Theorem (B. Rhoades)

The conjecture is true.
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Theorem (B. Rhoades)

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Definition
Given $0 < a < b$ coprime, if we define

$$\text{Ass}'(a, b) := \begin{cases} 
\text{Ass}(a, b - a) & \text{for } a < (b - a) \\
\text{Ass}(b - a, a) & \text{for } (b - a) < a 
\end{cases}$$

then the number of **homotopy spheres** of $\text{Ass}(a, b)$ equals the number of **maximal faces** of $\text{Ass}'(a, b)$.

Question
What does the following mean?

$$\text{Ass}(a, b) \mapsto \text{Ass}'(a, b) \mapsto \text{Ass}''(a, b) \mapsto \cdots \mapsto \text{a point}$$
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Euclidean Algorithm = ?

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Epilogue: Parking Functions
Definition

- Label the up-steps by \( \{1, 2, \ldots, a\} \), increasing up columns.

- Call this a parking function.

- Let \( \text{PF}(x) = \text{PF}(a, b) \) denote the set of parking functions.

- Classical form \((z_1, z_2, \ldots, z_a)\) has label \(z_i\) in column \(i\).

- Example: \((3, 1, 4, 4, 1)\)
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- Example: \((3, 1, 4, 4, 1)\)
The Rational Parking Space

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$

- By abuse, let $\text{PF}(x) = \text{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the rational parking space.
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- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

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- By abuse, let $PF(x) = PF(a, b)$ denote this representation of $\mathfrak{S}_a$.

- Call it the **rational parking space**.
The dimension of $PF(a, b)$ is $b^{a-1}$.

The complete homogeneous expansion is

$$PF(a, b) = \sum_{r \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_r,$$

where the sum is over $r = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

That is: $PF(a, b)$ is the coefficient of $t^a$ in $\frac{1}{b} H(t)^b$, where

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Theorems (with N. Loehr and G. Warrington)

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That is: PF($a, b$) is the coefficient of $t^a$ in $\frac{1}{b} H(t)^b$, where

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Then using the Cauchy product identity we get...

- The power sum expansion is

\[ \text{PF}(a, b) = \sum_{r \vdash a} b^{\ell(r)-1} \frac{p_r}{z_r} \]

i.e. the # of parking functions fixed by \( \sigma \in \mathfrak{S}_a \) is \( b^{\# \text{cycles}(\sigma)-1} \).

- The Schur expansion is

\[ \text{PF}(a, b) = \sum_{r \vdash a} \frac{1}{b^{s_r(1^b)}} s_r. \]
A Few Facts

Theorems (with N. Loehr and G. Warrington)

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PF(a, b) = \sum_{r \vdash a} \frac{1}{b} s_r(1^b) s_r.
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The multiplicities of the hook Schur functions $s[k + 1, 1^{a-k-1}]$ in $\text{PF}(a, b)$ are given by the rational Schröder numbers:

$$\text{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1, 1^{a-k-1}]}(1^b) = \frac{1}{b} \binom{a-1}{k} \binom{b+k}{a}.$$ 

Special Cases:

- Trivial character: $\text{Schrö}(a, b; a - 1) = \text{Cat}(a, b)$.

- Smallest $k$ that occurs is $k = \max\{0, a - b\}$, in which case

$$\text{Schrö}(a, b; k) = \text{Cat}'(a, b).$$

- Hence $\text{Schrö}(x; k)$ interpolates between $\text{Cat}(x)$ and $\text{Cat}'(x)$. 
A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** \( s[k + 1, 1^{a-k-1}] \) in \( \text{PF}(a, b) \) are given by the **rational Schröder numbers**:

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The multiplicities of the **hook Schur functions** $s[k + 1, 1^{a-k-1}]$ in $PF(a, b)$ are given by the **rational Schröder numbers**:

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- **Hence $Schrö(x; k)$ interpolates between $Cat(x)$ and $Cat'(x)$**.
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- Hence $\text{Schrö}(x; k)$ interpolates between $\text{Cat}(x)$ and $\text{Cat}'(x)$. 
What does switching $a \leftrightarrow b$ mean?

**Problem**

Given $a, b$ coprime we have an $\mathbb{S}_a$-module $\text{PF}(a, b)$ of dimension $b^{a-1}$ and an $\mathbb{S}_b$-module $\text{PF}(b, a)$ of dimension $a^{b-1}$.

- What is the relationship between $\text{PF}(a, b)$ and $\text{PF}(b, a)$?
- Note that hook multiplicities are the same:

  $$\text{Schrö}(a, b; k) = \text{Schrö}(b, a; k + b - a).$$

What does switching $a \leftrightarrow b$ mean?

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The Kirkman/Narayana/Schröder numbers are equivalent. They contain information about rank. \((1 < k < a − 1)\)

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\begin{align*}
\text{Kirk}(x; k) &= \frac{1}{a} \binom{a}{k} \binom{b+k-1}{k-1} & \text{f-vector} \\
\text{Nar}(x; k) &= \frac{1}{a} \binom{a}{k} \binom{b-1}{k-1} & \text{h-vector} \\
\text{Schrö}(x; k) &= \frac{1}{b} \binom{a-1}{k} \binom{b+k}{a} & \text{“dual” f-vector}
\end{align*}
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The Kreweras numbers are more refined. They contain parabolic information. \((r \vdash a)\)

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But what about $q$ and $t$?

There exists a bigraded version $PF_{q,t}(a, b)$. Here is the coefficient of the (non-hook) Schur function $s[2, 2, 1]$ in $PF_{q,t}(5, 8)$:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 3 & 4 & 3 & 2 & 1 \\
2 & 6 & 6 & 4 & 2 & 1 \\
2 & 7 & 7 & 4 & 2 & 1 \\
1 & 6 & 7 & 4 & 2 & 1 \\
3 & 6 & 4 & 2 & 1 \\
1 & 4 & 4 & 2 & 1 \\
1 & 3 & 2 & 1 \\
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1
\end{pmatrix}
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But what about $q$ and $t$?

Tease

There **exists** a bigraded version $\text{PF}_{q,t}(a, b)$. Here is the coefficient of the (non-hook) Schur function $s[2, 2, 1]$ in $\text{PF}_{q,t}(5, 8)$:

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1 & 2 & 1 \\
1 & 1 \\
1
\end{array}
$$
Thanks! Here is a crazy picture.

by Dan Drake and Drew Armstrong