The Yoneda Lemma. Given a category $\mathcal{C}$, there is one easy functor and one hard functor from the product category $\mathsf{Set}^{\mathcal{C}} \times \mathcal{C}$ into the category Set:

- Easy: Given a functor $F \in \mathsf{Set}^{\mathcal{C}}$ and an object $c \in \mathcal{C}$, the assignment $\mathsf{Eval}(F, c) := F(c)$ defines the *evaluation functor*

$$\mathsf{Eval} : \mathsf{Set}^{\mathcal{C}} \times \mathcal{C} \to \mathsf{Set}.$$

- Hard: Given an object $c \in \mathcal{C}$ recall that we have a functor $H^c = \mathsf{Hom}_{\mathcal{C}}(c, -) \in \mathsf{Set}^{\mathcal{C}}$. Then for any other functor $F \in \mathsf{Set}^{\mathcal{C}}$ we will use the notation $\mathsf{Nat}(H^c, F)$ for the set[1] of natural transformations $N^c \Rightarrow F$. The assignment $\mathsf{Yon}(F, c) := \mathsf{Nat}(H^c, F)$ defines the *Yoneda functor*

$$\mathsf{Yon} : \mathsf{Set}^{\mathcal{C}} \times \mathcal{C} \to \mathsf{Set}.$$

The Yoneda Lemma says that the family of functions $\Xi_{F,c} : \mathsf{Yon}(F, c) \to \mathsf{Eval}(F, c)$ defined by sending $\Phi \in \mathsf{Nat}(H^c, F)$ to $\Phi_c(\mathsf{id}_c) \in F(c)$ is a **natural isomorphism**

$$\Xi : \mathsf{Yon} \cong \mathsf{Eval}.$$

///

In summary, for each functor $F : \mathcal{C} \to \mathsf{Set}$ and object $c \in \mathcal{C}$, the Yoneda Lemma says that

"natural transformations $H^c \Rightarrow F$ are the same as elements of $F(c)$."

In particular, this implies that the collection $\mathsf{Nat}(H^c, F)$ is a **set**, which was not a priori obvious. There is also a dual version of the Yoneda Lemma which says that, for each functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ and each object $c \in \mathcal{C}$, the mapping $\Phi \mapsto \Phi_c(\mathsf{id}_c)$ defines a natural bijection $\mathsf{Nat}(H_c, F) \to F(c)$. To obtain a proof of the dual version, just "reverse all arrows" in the following proof.

Because of the unifying power of the Yoneda Lemma, you should expect that there will be a lot of details to check. Every mathematician should go through the details of this proof **exactly once** in their life. (Typing up this proof was my one time.)

Proof of Yoneda: As with many theorems of category theory, the real difficulty is in keeping track of the definitions. We will go very slowly.

(1) **Define the Functors.** We were only told how the functors Eval and Yon act on objects. We need to examine how they act on arrows. Since each functor is defined on the product category $\mathsf{Set}^{\mathcal{C}} \times \mathcal{C}$, it must act separately on arrows in $\mathsf{Set}^{\mathcal{C}}$ and in $\mathcal{C}$, and these two actions must commute.

First we look at Eval. For each arrow $\varphi : c_1 \to c_2$ in $\mathcal{C}$ and each functor $F \in \mathsf{Set}^{\mathcal{C}}$, we must find an arrow $\mathsf{Eval}(F, \varphi) : \mathsf{Eval}(F, c_1) \to \mathsf{Eval}(F, c_2)$ — that is, a function $\mathsf{Eval}(F, \varphi) : F(c_1) \to F(c_2)$ — with the property that $\mathsf{Eval}(F, \varphi \circ \psi) = \mathsf{Eval}(F, \varphi) \circ \mathsf{Eval}(F, \psi)$. This is easy; we just take $\mathsf{Eval}(F, \varphi) := F(\varphi) : F(c_1) \to F(c_2)$, which exists because F is a

[1] It is not a priori obvious that this collection of natural transformations is a set. The fact that it is a set will follow from the proof.

functor. Then for each object $c \in \mathcal{C}$ and each arrow $F_1 \Rightarrow F_2$ in $\mathsf{Set}^{\mathcal{C}}$, we must find an arrow $\mathsf{Eval}(\Phi, c) : \mathsf{Eval}(F_1, c) \to \mathsf{Eval}(F_2, c)$ — that is, a function $\mathsf{Eval}(\Phi, c) : F_1(c) \to F_2(c)$ — with the property that $\mathsf{Eval}(\Phi \circ \Psi, c) = \mathsf{Eval}(\Phi, c) \circ \mathsf{Eval}(\Psi, c)$. This is also easy; we just take $\mathsf{Eval}(\Phi, c) := \Phi_c : F_1(c) \to F_2(c)$, which exists because Φ is a natural transformation. Finally, the naturality of Φ says that the following square commutes:

$$
\begin{tikzcd}
F_1(c_2) \arrow[r, "\Phi_{c_2}"] & F_2(c_2) \\
F_1(c_1) \arrow[u, "F_1(\varphi)"] \arrow[ur, "\mathsf{Eval}(\Phi,\varphi)"] \arrow[r, "\Phi_{c_1}"'] & F_2(c_1) \arrow[u, "F_2(\varphi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Eval}(F_1, c_2) \arrow[r, "\mathsf{Eval}(\Phi,c_2)"] & \mathsf{Eval}(F_2, c_2) \\
\mathsf{Eval}(F_1, c_1) \arrow[u, "\mathsf{Eval}(F_1,\varphi)"] \arrow[ur, "\mathsf{Eval}(\Phi,\varphi)"] \arrow[r, "\mathsf{Eval}(\Phi,c_1)"'] & \mathsf{Eval}(F_2, c_1) \arrow[u, "\mathsf{Eval}(F_2,\varphi)"']
\end{tikzcd}
$$

Thus we can define the function $\mathsf{Eval}(\Phi, \varphi) : \mathsf{Eval}(F_1, c_1) \to \mathsf{Eval}(F_2, c_2)$ by following either path from the bottom left to the top right of the square. Explicitly, we have $\mathsf{Eval}(\Phi, \varphi) := F_2(\varphi) \circ \Phi_{c_1} = \Phi_{c_2} \circ F_2(\varphi)$.

Next we look at Yon. For each arrow $\varphi : c_1 \to c_2$ in $\mathcal{C}$ and each functor $F \in \mathsf{Set}^{\mathcal{C}}$, we must find an arrow $\mathsf{Yon}(F, \varphi) : \mathsf{Yon}(F, c_1) \to \mathsf{Yon}(F, c_2)$ — that is, a function $\mathsf{Yon}(F, \varphi) : \mathsf{Nat}(H^{c_1}, F) \to \mathsf{Nat}(H^{c_2}, F)$ — with the property that $\mathsf{Yon}(F, \varphi \circ \psi) = \mathsf{Yon}(F, \varphi) \circ \mathsf{Yon}(F, \psi)$. To define this, let $\Lambda \in \mathsf{Nat}(H^{c_1}, F)$ be any natural transformation, so that for each arrow $\lambda : d_1 \to d_2$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_1, d_2) \arrow[r, "\Lambda_{d_2}"] & F(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_1, d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "\Lambda_{d_1}"'] & F(d_1) \arrow[u, "F(\lambda)"']
\end{tikzcd}
$$

We can extend this diagram on the left using by φ:

$$
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_2, d_2) \arrow[r, "(-)\circ\varphi"] \arrow[rr, bend left, "\Lambda_{d_2}((-)\circ\varphi)"] & \mathsf{Hom}_{\mathcal{C}}(c_1, d_2) \arrow[r, "\Lambda_{d_2}"] & F(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_2, d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "(-)\circ\varphi"'] \arrow[rr, bend right, "\Lambda_{d_1}((-)\circ\varphi)"'] & \mathsf{Hom}_{\mathcal{C}}(c_1, d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "\Lambda_{d_2}"'] & F(d_1) \arrow[u, "F(\lambda)"']
\end{tikzcd}
$$

The new diagram commutes because λ and φ are acting on opposite sides. Then for each object $d \in \mathcal{C}$ we define the function $\mathsf{Yon}(F, \varphi)(\Lambda)_d : H^{c_2}(d) \to F(d)$ by $\mathsf{Yon}(F, \varphi)(\Lambda)_d(-) := \Lambda_d((-) \circ \varphi)$, and the above diagram says that these functions assemble into a natural transformation $\mathsf{Yon}(F, \varphi)(\Lambda) \in \mathsf{Nat}(H^{c_2}, F)$. The fact that $\mathsf{Yon}(F, \varphi \circ \psi) = \mathsf{Yon}(F, \varphi) \circ \mathsf{Yon}(F, \psi)$ follows by extending the diagram twice on the left and then using the associativity of composition.

On the other hand, for each arrow $\Phi : F_1 \Rightarrow F_2$ in $\mathsf{Set}^{\mathcal{C}}$ and each object $c \in \mathcal{C}$, we must find an arrow $\mathsf{Yon}(\Phi, c) : \mathsf{Yon}(F_1, c) \to \mathsf{Yon}(F_1, c)$ — that is, a function $\mathsf{Yon}(\Phi, c) : \mathsf{Nat}(H^c, F_1) \to \mathsf{Nat}(H^c, F_2)$ — with the property that $\mathsf{Yon}(\Phi \circ \Psi, c) = \mathsf{Yon}(\Phi, c) \circ \mathsf{Yon}(\Psi, c)$. To define this, let $\Lambda \in \mathsf{Nat}(H^c, F_1)$ be any natural transformation, so that for each arrow $\lambda : d_1 \to d_2$ in $\mathcal{C}$

the following diagram commutes:

$$
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,d_2) \arrow[r, "\Lambda_{d_2}"] & F_1(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c,d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "\Lambda_{d_2}"'] & F_1(d_1) \arrow[u, "F_1(\lambda)"']
\end{tikzcd}
$$

We can extend this diagram on the right by using Φ:

$$
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,d_2) \arrow[r, "\Lambda_{d_2}"] \arrow[rr, bend left, "\Phi_{d_2}\circ\Lambda_{d_2}"] & F_1(d_2) \arrow[r, "\Phi_{d_2}"] & F_2(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c,d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "\Lambda_{d_2}"'] \arrow[rr, bend right, "\Phi_{d_1}\circ\Lambda_{d_1}"'] & F_1(d_1) \arrow[u, "F_1(\lambda)"'] \arrow[r, "\Phi_{d_1}"'] & F_2(d_1) \arrow[u, "F_2(\lambda)"']
\end{tikzcd}
$$

The new diagram commutes because Φ is a natural transformation. Then for each object $d \in \mathcal{C}$ we define the function $\mathsf{Yon}(\Phi, c)(\Lambda)_d : H^c \to F_2$ by $\mathsf{Yon}(\Phi, c)(\Lambda)_d := \Phi_d \circ \Lambda_d$, and the above diagram says that these functions assemble into a natural transformation $\mathsf{Yon}(\Phi, c)(\Lambda) \in \mathsf{Nat}(H^c, F_2)$. The fact that $\mathsf{Yon}(\Phi \circ \Psi, c) = \mathsf{Yon}(\Phi, c) \circ \mathsf{Yon}(\Psi, c)$ follows by extending the diagram twice on the right and then using the associativity of composition. Finally, for each arrow $\varphi : c_1 \to c_2$ in $\mathcal{C}$ and each arrow $\Phi : F_1 \Rightarrow F_2$ in $\mathsf{Set}^{\mathcal{C}}$ we observe that the following diagram commutes:

$$
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_2,d_2) \arrow[r, "(-)\circ\varphi"] \arrow[rr, bend left, "\Lambda_{d_2}((-)\circ\varphi)"] \arrow[rrr, bend left=50, "\Phi_{d_2}(\Lambda_{d_2}((-)\circ\varphi))"] & \mathsf{Hom}_{\mathcal{C}}(c_1,d_2) \arrow[r, "\Lambda_{d_2}"] \arrow[rr, bend left, "\Phi_{d_2}\circ\Lambda_{d_2}"] & F_1(d_2) \arrow[r, "\Phi_{d_2}"] & F_2(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_2,d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "(-)\circ\varphi"'] \arrow[rr, bend right, "\Lambda_{d_1}((-)\circ\varphi)"'] \arrow[rrr, bend right=50, "\Phi_{d_1}(\Lambda_{d_1}((-)\circ\varphi))"'] & \mathsf{Hom}_{\mathcal{C}}(c_1,d_1) \arrow[u, "\lambda\circ(-)"] \arrow[r, "\Lambda_{d_2}"'] \arrow[rr, bend right, "\Phi_{d_1}\circ\Lambda_{d_1}"'] & F_1(d_1) \arrow[u, "F_1(\lambda)"'] \arrow[r, "\Phi_{d_1}"'] & F_2(d_1) \arrow[u, "F_2(\lambda)"']
\end{tikzcd}
$$

Thus for each object $d \in \mathcal{C}$ and natural transformation $\Lambda : H^{c_1} \Rightarrow F_1$ we define a function $\mathsf{Yon}(\Phi, \varphi)(\Lambda)_d(-) : \Phi_d(\Lambda_d((-) \circ \varphi))$, and the above diagram says that these functions assmemble into a natural transformation $\mathsf{Yon}(\Phi, \varphi)(\Lambda) : H^{c_2} \Rightarrow F_2$. In summary, we have obtained a function $\mathsf{Yon}(\Phi,\varphi):\mathsf{Yon}(F_1,c_1)\to\mathsf{Yon}(F_2,c_2)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Nat}(H^{c_2},F_1) \arrow[r, "\Phi\circ(-)"] & \mathsf{Nat}(H^{c_2},F_2) \\
\mathsf{Nat}(H^{c_1},F_1) \arrow[u, "(-)((-)\circ\varphi)"] \arrow[r, "\Phi\circ(-)"'] \arrow[ur, "\mathsf{Yon}(\Phi,\varphi)"] & \mathsf{Nat}(H^{c_1},F_2) \arrow[u, "(-)((-)\circ\varphi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Yon}(F_1,c_2) \arrow[r, "\mathsf{Yon}(\Phi,c_2)"] & \mathsf{Yon}(F_2,c_2) \\
\mathsf{Yon}(F_1,c_1) \arrow[u, "\mathsf{Yon}(F_1,\varphi)"] \arrow[r, "\mathsf{Yon}(\Phi,c_1)"'] \arrow[ur, "\mathsf{Yon}(\Phi,\varphi)"] & \mathsf{Yon}(F_2,c_1) \arrow[u, "\mathsf{Yon}(F_2,\varphi)"']
\end{tikzcd}$$

Now at least we know what we are supposd to prove.

(2) **Naturality of Ξ.** For each functor $F\in\mathsf{Set}^{\mathcal{C}}$ and each object $c\in\mathcal{C}$ we haved defined the function $\Xi_{F,c}:\mathsf{Nat}(H^c,F)\to F(c)$ by sending each natural transformation $\Phi:H^c\Rightarrow F$ to the element $\Phi_c(\mathsf{id}_c)\in F(c)$. We want to show that the functions $\Xi_{F,c}$ assemble into a natural transformation $\Xi:\mathsf{Yon}\Rightarrow\mathsf{Eval}$. Since $\mathsf{Yon},\mathsf{Eval}:\mathsf{Set}^{\mathcal{C}}\times\mathcal{C}\to\mathsf{Set}$ are "bifunctors", this amounts to proving two separate statements:

- For each fixed $F\in\mathsf{Set}^{\mathcal{C}}$ the functions $\Xi_{F,c}$ assemble into a natural transformation
$$\Xi_{F,-}:\mathsf{Nat}(H^{(-)},F)\Rightarrow F(-).$$
- For each fixed $c\in\mathcal{C}$ the functions $\Xi_{F,c}$ assemble into a natural transformation
$$\Xi_{-,c}:\mathsf{Nat}(H^c,-)\Rightarrow(-)(c).$$

To prove the first statement, let $\varphi:c_1\to c_2$ be any arrow in $\mathcal{C}$. For each fixed $F\in\mathsf{Set}^{\mathcal{C}}$ we need to show that the following square commutes:

$$\begin{tikzcd}
\mathsf{Nat}(H^{c_2},F) \arrow[r, "(-)_{c_2}(\mathsf{id}_{c_2})"] & F(c_2) \\
\mathsf{Nat}(H^{c_1},F) \arrow[u, "(-)((-)\circ\varphi)"] \arrow[r, "(-)_{c_1}(\mathsf{id}_{c_1})"'] & F(c_1) \arrow[u, "F(\varphi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Yon}(F,c_2) \arrow[r, "\Xi_{F,c_2}"] & \mathsf{Eval}(F,c_2) \\
\mathsf{Yon}(F,c_1) \arrow[u, "\mathsf{Yon}(F,\varphi)"] \arrow[r, "\Xi_{F,c_1}"'] & \mathsf{Eval}(F,c_1) \arrow[u, "\mathsf{Eval}(F,\varphi)"']
\end{tikzcd}$$

To show this, consider any natural transformation $\Lambda\in\mathsf{Nat}(H^{c_1},F)=\mathsf{Yon}(F,c_1)$. Then following Λ around the bottom of the square gives

$$(\mathsf{Eval}(F,\varphi)\circ\Xi_{F,c_1})(\Lambda)=F(\varphi)(\Lambda_{c_1}(\mathsf{id}_{c_1}))$$

and following Λ around the top of the square gives

$$(\Xi_{F,c_2}\circ\mathsf{Yon}(F,\varphi))(\Lambda)=\Lambda_{c_2}(\mathsf{id}_{c_1}\circ\varphi)=\Lambda_{c_2}(\varphi).$$

But since $\Lambda:H^{c_1}\Rightarrow F$ is a natural transformation the following square must commute:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_1,c_2) \arrow[r, "\Lambda_{c_2}"] & F(c_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_1,c_1) \arrow[u, "\varphi\circ(-)"] \arrow[r, "\Lambda_{c_1}"'] & F(c_1) \arrow[u, "F(\varphi)"']
\end{tikzcd}$$

Finally, by following the element $\mathsf{id}_{c_1} \in \mathsf{Hom}_{\mathcal{C}}(c_1, c_1)$ from the bottom left to the top right in both ways, we obtain

$$\begin{aligned}(F(\varphi) \circ \Lambda_{c_1}(\mathsf{id}_{c_1})) &= \Lambda_{c_2}(\varphi \circ \mathsf{id}_{c_1})\\ F(\varphi)(\Lambda_{c_1}(\mathsf{id}_{c_1})) &= \Lambda_{c_2}(\varphi),\end{aligned}$$

as desired.

To prove the second statement, let $\Phi : F_1 \Rightarrow F_2$ be any arrow in $\mathsf{Set}^{\mathcal{C}}$. For each fixed $c \in \mathcal{C}$ we need to show that the following square commutes:

$$\begin{tikzcd}
\mathsf{Nat}(H^c, F_2) \arrow[r, "(-)_c(\mathsf{id}_c)"] & F_2(c) \\
\mathsf{Nat}(H^c, F_1) \arrow[u, "\Phi\circ(-)"] \arrow[r, "(-)_c(\mathsf{id}_c)"'] & F_1(c) \arrow[u, "\Phi_c"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Yon}(F_2, c) \arrow[r, "\Xi_{F_2,c}"] & \mathsf{Eval}(F_2, c) \\
\mathsf{Yon}(F_1, c) \arrow[u, "\mathsf{Yon}(\Phi,c)"] \arrow[r, "\Xi_{F_1,c}"'] & \mathsf{Eval}(F_1, c) \arrow[u, "\mathsf{Eval}(\Phi,c)"']
\end{tikzcd}$$

To show this, consider any natural transformation $\Lambda \in \mathsf{Nat}(H^c, F_1) = \mathsf{Yon}(F_1, c)$. Then following Λ around the bottom of the square gives

$$(\mathsf{Eval}(\Phi, c) \circ \Xi_{F_1,c})(\Lambda) = \Phi_c(\Lambda_c(\mathsf{id}_c))$$

and following Λ around the top of the square gives

$$(\Xi_{F_2,c} \circ \mathsf{Yon}(\Phi, c))(\Lambda) = (\Phi \circ \Lambda)_c(\mathsf{id}_c).$$

But recall that the composition of natural transformations is defined by $(\Phi \circ \Lambda)_c := \Phi_c \circ \Lambda_c$ for all $c \in \mathcal{C}$, and hence we have

$$(\Phi_c \circ \Lambda_c)(\mathsf{id}_c) = \Phi_c(\Lambda_c(\mathsf{id}_c)),$$

as desired.

(3) **Invertibility of Ξ.** It remains to show that for each functor $F \in \mathsf{Set}^{\mathcal{C}}$ and each object $c \in \mathcal{C}$ the function $\Xi_{F,c} : \mathsf{Yon}(F, c) \to \mathsf{Eval}(F, c)$ is invertible, and hence that $\Xi : \mathsf{Yon} \cong \mathsf{Eval}$ is a natural isomorphism of functors.

To define the inverse function $\Xi^{-1}_{F,c} : \mathsf{Eval}(F, c) \to \mathsf{Yon}(F, c)$, we must send each set element $x \in F(c) = \mathsf{Eval}(F, c)$ to a natural transformation $\Xi^{-1}_{F,c}(x) \in \mathsf{Nat}(H^c, F) = \mathsf{Yon}(F, c)$. And since H^c and F are functors $\mathcal{C} \to \mathsf{Set}$, this natural transformation $\Xi^{-1}_{F,c}(x) : H^c \Rightarrow F$ consists of a family of functions $\Xi^{-1}_{F,c}(x)_d : H^c(d) \to F(d)$, one for each object $d \in \mathcal{C}$. That is, for each arrow $\varphi \in \mathsf{Hom}_{\mathcal{C}}(c, d) = H^c(d)$ we must define a set element $\Xi^{-1}_{F,c}(x)_d(\varphi) \in F(d)$. Applying the functor F to the arrow $\varphi : c \to d$ yields a function $F(\varphi) : F(c) \to F(d)$ and thus we can make the following definition:

$$\Xi^{-1}_{F,c}(x)_d(\varphi) := F(\varphi)(x) \in F(d).$$

Now we need to check two things:

- The functions $\Xi^{-1}_{F,c}(x)_d$ assemble into a natural transformation $\Xi^{-1}_{F,c}(x) : H^c \Rightarrow F$, and hence we obtain a function $\Xi^{-1}_{F,c} : \mathsf{Eval}(F, c) \to \mathsf{Yon}(F, c)$.
- The functions $\Xi_{F,c}$ and $\Xi^{-1}_{F,c}$ are inverse.

To check the first statement, let $\lambda : d_1 \to d_2$ be any arrow in $\mathcal{C}$. We must show that the following square commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,d_2) \arrow[r, "F(-)(x)"] & F(d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c,d_1) \arrow[r, "F(-)(x)"'] \arrow[u, "\lambda\circ(-)"] & F(d_1) \arrow[u, "F(\lambda)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^c(d_2) \arrow[r, "\Xi^{-1}_{F,c}(x)_{d_2}"] & F(d_2) \\
H^c(d_1) \arrow[r, "\Xi^{-1}_{F,c}(x)_{d_1}"'] \arrow[u, "H^c(\lambda)"] & F(d_1) \arrow[u, "F(\lambda)"']
\end{tikzcd}$$

And to see this, consider any arrow $\varphi \in \mathsf{Hom}_{\mathcal{C}}(c, d_1)$. Following φ around the bottom of the square gives $F(\lambda)(F(\varphi)(x))$ and following φ around the top of the square gives $F(\lambda \circ \varphi)(x)$, which equals

$$F(\lambda \circ \varphi)(x) = (F(\lambda) \circ F(\varphi))(x) = F(\lambda)(F(\varphi)(x)),$$

by the functoriality of F.

To check the second statement we will first show that $\Xi_{F,c} \circ \Xi^{-1}_{F,c}$ is the identity function on $\mathsf{Eval}(F, c)$. So consider any set element $x \in F(c) = \mathsf{Eval}(F, c)$. Then by the definitions of $\Xi_{F,c}$ and $\Xi^{-1}_{F,c}$ and by the functoriality of F we have

$$\begin{aligned}
(\Xi_{F,c} \circ \Xi^{-1}_{F,c})(x) &= \Xi_{F,c}(\Xi^{-1}_{F,c}(x)) && \\
&= \Xi^{-1}_{F,c}(x)_c(\mathsf{id}_c) && \text{definition of } \Xi_{F,c} \\
&= F(\mathsf{id}_c)(x) && \text{definition of } \Xi^{-1}_{F,c} \\
&= \mathsf{id}_{F(c)}(x) && \text{functoriality of } F \\
&= x.
\end{aligned}$$

Finally, we will show that $\Xi^{-1}_{F,c} \circ \Xi_{F,c}$ is the identity function on $\mathsf{Yon}(F, c)$. So consider any natural transformation $\Phi \in \mathsf{Nat}(H^c, F) = \mathsf{Yon}(F, c)$. For any object $d \in \mathcal{C}$ we want to show that Φ_d and $(\Xi^{-1}_{F,c} \circ \Xi_{F,c})(\Phi)_d$ define the same function $H^c(d) \to F(d)$. So consider any arrow $\varphi \in \mathsf{Hom}_{\mathcal{C}}(c, d) = H^c(d)$. The naturality of Φ says that the following square commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,d) \arrow[r, "\Phi_d"] & F(d) \\
\mathsf{Hom}_{\mathcal{C}}(c,c) \arrow[r, "\Phi_c"'] \arrow[u, "\varphi\circ(-)"] & F(c) \arrow[u, "F(\varphi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^c(d) \arrow[r, "\Phi_d"] & F(d) \\
H^c(c) \arrow[r, "\Phi_c"'] \arrow[u, "H^c(\varphi)"] & F(c) \arrow[u, "F(\varphi)"']
\end{tikzcd}$$

In particular, following the arrow id_c from the bottom left to the top right in two ways gives

$$F(\varphi)(\Phi_c(\mathsf{id}_c)) = \Phi_d(\varphi \circ \mathsf{id}_c) = \Phi_d(\varphi).$$

Then by the definitions of $\Xi_{F,c}$ and $\Xi^{-1}_{F,c}$ and by the naturality of Φ we have

$$\begin{aligned}
(\Xi^{-1}_{F,c} \circ \Xi_{F,c})(\Phi)_d(\varphi) &= \Xi^{-1}_{F,c}(\Xi_{F,c}(\Phi))_d(\varphi) && \\
&= \Xi^{-1}_{F,c}(\Phi_c(\mathsf{id}_c))_d(\varphi) && \text{definition of } \Xi_{F,c} \\
&= F(\varphi)(\Phi_c(\mathsf{id}_c)) && \text{definition of } \Xi^{-1}_{F,c} \\
&= \Phi_d(\varphi), && \text{naturality of } \Phi
\end{aligned}$$

as desired.

This completes the proof of Yoneda's Lemma. ///