Noncommutative Algebra — Fall 2016
Exercises 6 — Drew Armstrong

6.1. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories and consider three bifunctors

$$F_1, F_2, F_3 : \mathcal{B} \times \mathcal{C} \to \mathcal{D}.$$

If $\Phi : F_1 \overset{\sim}{\Rightarrow} F_2$ and $\Psi : F_2 \overset{\sim}{\Rightarrow} F_3$ are natural isomorphisms of bifunctors, prove that the composition $\Psi \circ \Phi$ is a natural isomorphism $F_1 \cong F_3$.

6.2. Consider two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ and suppose that we have a natural transformation

$$\Phi : F_1 \Rightarrow F_2.$$

If each of the arrows $\Phi_c : F_1(c) \to F_2(c)$ is invertible, prove that the inverses $\Phi_c^{-1} : F_2(c) \to F_1(c)$ assemble into a natural transformation $\Phi^{-1} : F_2 \Rightarrow F_1$, and hence we have a natural isomorphism $F_1 \cong F_2$.

6.3. Fix a small category $\mathcal{I}$ and recall that for any object $c \in \mathcal{C}$ we have a *constant diagram* $c^{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$ that sends each object in $\mathcal{I}$ to c and each arrow in $\mathcal{I}$ to id_c. Prove that:

(a) For any category $\mathcal{C}$ the mapping $c \mapsto c^{\mathcal{I}}$ defines a functor $(-)^{\mathcal{I}} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ which we call the *diagonal functor*.

(b) For any functor $F : \mathcal{C} \to \mathcal{D}$ the mapping $D \mapsto F(D) := F \circ D$ defines a functor $F : \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ with the property that $F(-)^{\mathcal{I}} = F((-)^{\mathcal{I}})$.

(c) Prove that any adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ determines a natural isomorphism

$$\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}}, R(-)) \cong \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L((-)^{\mathcal{I}}), -)$$

of bifunctors $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}^{\mathcal{I}} \to \mathsf{Set}$.