Noncommutative Algebra **Fall 2016**
Exercises 5 **Drew Armstrong**

5.1. I claimed in the notes that the direct product of groups is the categorical product in Grp. Check that this is true.

5.2. I claimed in the notes that a ring is a ringoid (i.e., a small Ab-category) with one object. Check that this agrees with the usual definition of rings.

5.3. Let R be a **commutative** ring and consider the category R-Mod of left R-modules.

(a) State the definitions of R-Mod-*category* and R-Mod-*functor*.

(b) Verify that R-Mod is an R-Mod-category.

(c) Try to guess definitions for *R-algebroid*, *R-algebra*, and *R-algebra representation*.

5.4. Let $\mathcal{C} \subseteq \mathsf{Set}$ be a concrete category and consider any arrow $\varphi : X \to Y$ in $\mathcal{C}$.

(a) Copy the proof from the case of $\mathcal{C} = \mathsf{Set}$ to show that

$$\begin{aligned} \varphi \text{ is injective} \quad &\Longrightarrow \quad \varphi \text{ is monic,} \\ \varphi \text{ is surjective} \quad &\Longrightarrow \quad \varphi \text{ is epic.} \end{aligned}$$

(b) Let $U : \mathcal{C} \to \mathsf{Set}$ be the "forgetful functor" that assigns to each object $X \in \mathcal{C}$ its underlying set $U(X) \in \mathsf{Set}$, and assume that U has a left adjoint "free functor" $F : \mathsf{Set} \to \mathcal{C}$. Use the "free object" $F(\{*\}) \in \mathcal{C}$ to prove that

$$\varphi \text{ is monic} \quad \Longrightarrow \quad \varphi \text{ is injective.}$$

(c) It is quite common for epic arrows to be non-surjective. For example, prove that the inclusion ring homomorphism $\iota : \mathbb{Z} \to \mathbb{Q}$ is epic even though it is clearly not surjective. The proof from the case $\mathcal{C} = \mathsf{Set}$ doesn't generalize because the category of rings doesn't have a "subobject classifier" such as $\Omega = \{0, 1\}$.