Noncommutative Algebra — Fall 2016
Exercises 4 — Drew Armstrong

4.1. Consider the poset categories of integers $(\mathbb{Z}, \leq)$ and real numbers $(\mathbb{R}, \leq)$ and let $\iota : \mathbb{Z} \to \mathbb{R}$ be the inclusion functor. Prove that ι has both a left and a right adjoint functor. What does the RAPL theorem say in this case?

4.2. Given $X, Y \in \mathsf{Set}$ we will write $Y^X := \mathsf{Hom}_{\mathsf{Set}}(X, Y)$ for the set of functions $X \to Y$.

(a) Given $X \in \mathsf{Set}$, prove that taking the Cartesian product with X defines a functor $(-) \times X : \mathsf{Set} \to \mathsf{Set}$.

(b) Prove that the functor $(-) \times X$ is left adjoint to the Hom functor $(-)^X : \mathsf{Set} \to \mathsf{Set}$.

(c) What does the RAPL theorem tell you in this case?

4.3. Given abelian groups $A, B \in \mathsf{Ab}$, recall that the Hom set $\mathsf{Hom}_{\mathsf{Ab}}(A, B)$ has the structure of an abelian group with addition defined pointwise:

$$(\varphi_1 + \varphi_2)(a) := \varphi_1(a) + \varphi_2(a) \quad \text{for all } a \in A.$$

(a) Given $A \in \mathsf{Ab}$ prove that we have a Hom functor $H^A := \mathsf{Hom}_{\mathsf{Ab}}(A, -) : \mathsf{Ab} \to \mathsf{Ab}$.

(b) Given $A \in \mathsf{Ab}$ prove that the direct sum with A defines a functor $(-) \oplus A : \mathsf{Ab} \to \mathsf{Ab}$, but this functor is **not** left adjoint to H^A. [Hint: It doesn't preserve colimits.]

(c) Nevertheless, I claim that the functor H^A **does** have a left adjoint:

$$(-) \otimes A : \mathsf{Ab} \rightleftarrows \mathsf{Ab} : H^A.$$

Without knowing the definition of $\otimes$, use the general properties of adjunctions to learn as much as you can.