Noncommutative Algebra **Fall 2016**
Exercises 2 **Drew Armstrong**

2.1. Let $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$ be a Galois connection. Prove that for all subsets $S \subseteq \mathcal{P}$ and $T \subseteq \mathcal{Q}$ we have

$$(\vee_{\mathcal{P}} S)^* = \wedge_{\mathcal{Q}} S^* \quad \text{and} \quad (\vee_{\mathcal{Q}} T)^* = \wedge_{\mathcal{P}} T^*.$$

2.2. Continuing from 2.1, prove that for all $S \subseteq \mathcal{P}$ and $T \subseteq \mathcal{Q}$ we have

$$\vee_{\mathcal{Q}} S^* \leq_{\mathcal{Q}} (\wedge_{\mathcal{P}} S)^* \quad \text{and} \quad \vee_{\mathcal{P}} T^* \leq_{\mathcal{P}} (\wedge_{\mathcal{Q}} T)^*.$$

2.3. Consider the real line $\mathbb{R}$ with the usual topology, and let $- : 2^{\mathbb{R}} \rightleftarrows 2^{\mathbb{R}} : \circ$ be the topological *closure* and *interior* operators. Let $\mathscr{O} \subseteq 2^{\mathbb{R}}$ and $\mathscr{C} \subseteq 2^{\mathbb{R}}$ denote the collections of *open* and *closed* subsets of $\mathbb{R}$, respectively.

(a) Prove that we have an adjunction $- : \mathscr{O} \rightleftarrows \mathscr{C} : \circ$.

(b) Find specific examples in which the inequalities of Exercise 2.2 are strict. That is, find two open sets $O_1, O_2 \subseteq \mathbb{R}$ such that $(O_1 \cap O_2)^- \subsetneq O_1^- \cap O_2^-$ and two closed sets $C_1, C_2 \subseteq \mathbb{R}$ such that $C_1^\circ \cup C_2^\circ \subsetneq (C_1 \cup C_2)^\circ$.

2.4. Let U, V be sets and let $* : 2^U \rightleftarrows 2^V : *$ be an arbitrary Galois connection. Prove that there exists a unique relation $\sim \subseteq U \times V$ such that for all $S \in 2^U$ and $T \in 2^V$ we have

$$S^* = \{v \in V : \forall s \in S, s \sim v\},$$
$$T^* = \{u \in U : \forall t \in T, u \sim t\}.$$

2.5. Let U, V be sets and let $L : 2^U \rightleftarrows 2^V : R$ be an arbitrary adjunction. Prove that there exists a unique function $f : U \to 2^V$ such that for all $S \in 2^U$ and $T \in 2^V$ we have

$$L(S) = \cup_{s \in S} f(s),$$
$$R(S) = \{u \in U : f(u) \subseteq T\}.$$