Noncommutative Algebra — Exercises 1

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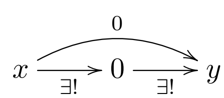
1.1. Let $\mathcal{C}$ be a category. Prove that $\mathcal{C}$-isomorphism is an equivalence relation on $\mathsf{Obj}(\mathcal{C})$.

1.2. Explain how a poset is the same thing as a small category $\mathcal{P}$ in which for all objects $x, y \in \mathcal{P}$ we have $|\mathsf{Hom}_{\mathcal{P}}(x,y)| \in \{0,1\}$.

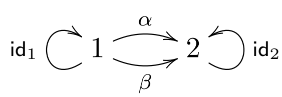
1.3. Let $\mathcal{P}$ be a poset. Show that if $\mathcal{P}$ contains arbitrary meets/joins, then it also contains arbitrary joins/meets.

1.4. Prove that a limit/colimit in a category is **unique**, in an appropriate sense.

1.5. Kernel/Cokernel. Let $\mathcal{C}$ be a category with a trivial object $0 \in \mathcal{C}$, i.e., an object that is both initial and final. Then for all objects $x, y \in \mathcal{C}$ we define the *trivial arrow* $0 : x \to y$ so the following diagram commutes:



Now let $\mathcal{I}$ be the category with two objects $\{1,2\}$ and four arrows as in the following picture:



For any arrow $\varphi : x \to y$ in $\mathcal{C}$ consider the diagram $D : \mathcal{I} \to \mathcal{C}$ defined by $D(1) := x$, $D(2) := y$, $D(\alpha) = \varphi$, and $D(\beta) = 0$. If the limit of D exists then we call it the *kernel of* φ and if the colimit exists we call it the *cokernel of* φ.

- Draw a picture summarizing the universal property of the kernel.
- Do the same for the cokernel.
- Prove that kernels exist in the category of groups. [Hint: Let $\varphi : G \to H$ be a homomorphism of groups and consider the inclusion homomorphism $\iota : \ker\varphi \to G$ from the set-theoretic kernel.]
- Prove that cokernels exist in the category of **abelian** groups. [Hint: Let $\varphi : A \to B$ be a group homomorphism and consider the quotient homomorphism $\pi : H \to H/\operatorname{im}\varphi$ by the set-theoretic image.]