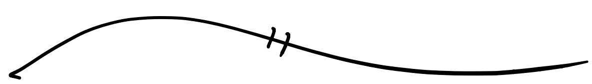
Current topic: Taylor series expansion and local behavior of curves (more generally, hypersurfaces).

Idea: Any polynomial in $R[x_1, \dots, x_n]$ has a Taylor expansion near any point $\vec{a} = (a_1, a_2, \dots, a_n) \in R^n$:

$$f(\vec{x}) = \sum_{I \in \mathbb{N}^n} b_I (\vec{x} - \vec{a})^I,$$

where the coefficients $b_I \in R$ are determined by formal derivatives.

[Formal means NO TOPOLOGY.]



For a variable x over a ring R, we define the R-linear function $D_x : R[x] \to R[x]$ by

$$D_x(x^k) = \begin{cases} k x^{k-1} & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases}$$

Problem 1.7: For all $f(x), g(x) \in R[x]$,

(a) $D_x(fg) = D_x(f)g + f D_x(g)$,

(b) $D_x(g^k) = k g^{k-1} D_x(g)$,

(c) $D_x(f \circ g) = [D_x(f) \circ g] D_x(g)$.

<u>Proof</u>:

(a): Check that $\Phi(f,g) = D_x(fg)$ & $\Psi(f,g) = D_x(f)g + f D_x(g)$ are R-bilinear functions $R[x]^2 \to R[x]$.

So it suffices to check identity on bases of monomials.

Let $f = x^m$, $g = x^n$.

$$
\begin{aligned}
D_x(f)g + f D_x(g) &= m x^{m-1} x^n + x^m n x^{n-1} \\
&= (m+n) x^{m+n-1} \\
&= D_x(x^{m+n}) = D_x(fg). \quad ///
\end{aligned}
$$

(b): $D_x(g^k) = k g^{k-1} D_x(g)$.

True for $k=1$ (and $k=0$).

Assume for some $k \geq 1$. Then

$$D_x(g^{k+1}) = D_x(g g^k)$$

$$= D_x(g) g^k + g k g^{k-1} D_x(g)$$

$$= (k+1) g^k D_x(g).$$

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(c): Given $f(x) = \sum a_k x^k$, we define the formal composition

$$(f \circ g)(\vec{x}) := \sum a_k (g(x))^k.$$

Then from linearity & part (b):

$$D_x(f \circ g) = \sum a_k D_x(g^k)$$

$$= \sum a_k k g^{k-1} D_x(g)$$

$$= [D_x(f) \circ g] D_x(g).$$

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Now we extend to partial derivatives.
Given $\vec{x} = \{x_1, \ldots, x_n\}$ variables $/R$, we define $D_{x_i} : R[\vec{x}] \to R[\vec{x}]$ by thinking

$$R[\vec{x}] = \underbrace{R[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]}_{R'}[x_i]$$

Since D_{x_i} is R'-linear & since $R \subseteq R'$ is a subring, then D_{x_i} is also R-linear. On the R-basis of monomials, we have

$$D_{x_i}(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n})$$
$$= k_i \cdot x_1^{k_1} \cdots x_i^{k_i - 1} \cdots x_n^{k_n}.$$

We observe that "mixed partials commute": For all $i \neq j$ and for all monomials $\vec{x}^K$ we have

$$D_{x_i} D_{x_j}(\vec{x}^{k}) = D_{x_j} D_{x_i}(\vec{x}^{k})$$

$$= k_i k_j \, x_1^{k_1} \cdots x_i^{k_i - 1} \cdots x_j^{k_j - 1} \cdots x_n^{k_n}.$$

By linearity this holds for all polynomials. ///

[Remark: This is much easier than Clairaut's Theorem!]

We have an important Theorem relating partial derivatives to projective equivalence & more general changes of coordinates.

<u>The Chain Rule</u>:

Let $\vec{x} = \{x_1, \ldots, x_n\}$ be independent variables over a ring R. Then for any polynomial $f(\vec{x}) \in R[\vec{x}]$ we define the <u>gradient row vector</u>

$$\nabla f = [D_{x_1} f, \ldots, D_{x_n} f] \in R[\vec{x}]^n.$$

For any vector of polynomials

$$\Phi = [\Phi_1(\vec{x}), \ldots, \Phi_n(\vec{x})] \in R[\vec{x}]^n$$

we define the formal composition

$$(f \circ \Phi)(\vec{x}) = f(\Phi_1(\vec{x}), \ldots, \Phi_n(\vec{x})) \in R[\vec{x}].$$

In this case, I claim that

$$\underbrace{\nabla(f \circ \Phi)}_{\text{row}} = \underbrace{[(\nabla f) \circ \Phi]}_{\text{row}} \cdot \underbrace{J\Phi}_{\text{matrix}}$$

where

$$J\Phi = (D_{x_j} \Phi_i) \in R[\vec{x}]^{n \times n}$$

is the Jacobian matrix of partial derivatives.

Example: If Φ is a <u>linear substitution</u>, i.e., if

$$\Phi(\vec{x}) = A\vec{x}$$

for some matrix $A \in \mathbb{R}^{n \times n}$, then

$$J\Phi = A.$$

[A linear function is equal to it's own linearization.]

<u>Proof</u>: What needs to be proved?

We need to show that

$$\begin{array}{c} j\text{th entry} \\ \text{of } \nabla(f \circ \Phi) \end{array} = \underset{\text{row}}{\left[(\nabla f) \circ \Phi\right]} \cdot \underset{\text{column}}{\left(j\text{th col } J\Phi\right)}.$$

By definition:

$$\begin{array}{c} i\text{th entry} \\ \text{of } \nabla(f) \circ \Phi \end{array} = (D_{x_i} f) \circ \Phi$$

$$= (D_{x_i} f)(\Phi_1(\vec{x}), \ldots, \Phi_n(\vec{x})).$$

If $f(\vec{x}) = \sum b_k \vec{x}^k$, then

$$D_{x_i} f = \sum b_k k_j \, x_1^{k_1} \cdots x_i^{k_i - 1} \cdots x_n^{k_n},$$

and hence

$$\begin{array}{l} i\text{th entry} \\ \text{of } \nabla(f)\circ\Phi \end{array} = \sum b_K k_i \Phi_1^{k_1} \cdots \Phi_i^{k_i - 1} \cdots \Phi_n^{k_n}.$$

On the other hand, the j^{th} entry of $\nabla(f\circ\Phi)$ is

$$D_{x_j}(f\circ\Phi) = D_{x_j} \sum b_K \Phi_1^{k_1} \cdots \Phi_n^{k_n}$$

$$= \sum b_K D_{x_j} \Phi_1^{k_1} \cdots \Phi_n^{k_n}$$

$$= \sum_K b_K \sum_{i=1}^{n} k_i \Phi_1^{k_1} \cdots \Phi_i^{k_i - 1} \cdots \Phi_n^{k_n} D_{x_j}\Phi_i$$

$$= \sum_i \left(\sum_K b_K k_i \Phi_1^{k_1} \cdots \Phi_i^{k_i - 1} \cdots \Phi_n^{k_n} \right) D_{x_j}\Phi_i$$

$$= \sum_i (i\text{th entry } \nabla f\circ\Phi)(i,j\text{ entry } J\Phi)$$

$$= [(\nabla f)\circ\Phi](j\text{th col of } J\Phi).$$

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