HW 2 due Mon Oct 12.

Notational Issue: GCD in UFD.

Let R be UFD. For any $a_1, \dots, a_n \in R$, the notation

$$\gcd(a_1, a_2, \dots, a_n) \text{ "=" } d.$$

What does it mean?

Theorem: There is a unique smallest prinical ideal I such that

$$a_1 R + a_2 R + \dots + a_n R \subseteq I \subseteq R.$$

Any generator of I is called a gcd of $a_1, \dots, a_n$.

<u>Proof</u>: Say $a_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots$

Then let $d = p_1^{\delta_1} p_2^{\delta_2} \cdots$

where $\delta_j = \min_i \{\alpha_{ij}\}$.

Have $d \mid a_i \ \forall i$

$\Rightarrow a_i R \subseteq dR \quad \forall i$

$\Longrightarrow \quad a_1R + \dots + a_nR \subseteq dR.$

If $a_iR \subseteq d'R \quad \forall i$ then

$d' \mid a_i$ for all i. By UFD,

$d' \mid d$, hence $dR \subseteq d'R$. ///

We could say:

$\gcd(a_1, \dots, a_n) =$ this smallest principal ideal.

For HW2, we always ignore units in the smallest ring, R.

Warning: UFD $\Rightarrow$ PID.

e.g. $\mathbb{F}[x,y]$ is not PID.

$x\mathbb{F}[x,y] + y\mathbb{F}[x,y]$ is not principal.

However, the elements x, y are coprime for reasons of degree.

$\gcd(x,y) = \mathbb{F}[x,y].$

In a PID we get something <u>more</u>.

Say $\gcd(a_1, \dots, a_n) = d$ in PID R.

Then we have <u>equality</u>

$$a_1 R + \cdots + a_n R = dR.$$

It follows that $\exists\ b_1, \dots, b_n$:

$$\boxed{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = d}$$

Bézout's Identity.

This is used in 3(a) & 4(b).

How?

$$\underset{\cancel{\text{PID}}}{\mathbb{F}[x,y]} = \mathbb{F}[x][y] \subseteq \underset{\text{PID } \checkmark}{\mathbb{F}(x)[y]}$$

where $\mathbb{F}(x) = \operatorname{Frac} \mathbb{F}[x]$.

Generalization:

$$\mathbb{F}[x,y][z] \subseteq \mathbb{F}(x,y)[z]$$

"covering spaces"

Last Time we defined projective space over a field:

$$\mathbb{FP}^n := \mathbb{F}^n \cup (\text{hyperplane at } \infty)$$

Any hyperplane could be the hyp. at ∞.

Today: Projective Subspaces.

A d-dimensional projective subspace of $\mathbb{FP}^n$ has the form

$$\mathbb{P}(V) = V/\sim$$

where $V \subseteq \mathbb{F}^{n+1}$ is a <u>linear</u> subspace of dimension $d+1$.

$$\begin{array}{c} d\text{-dim proj} \\ \text{subsp } \mathbb{FP}^n \end{array} \longleftrightarrow \begin{array}{c} d+1\text{-dim linear} \\ \text{sub sp. } \mathbb{F}^{n+1} \end{array}$$

Convention: $\mathbb{P}(\{0\}) = \emptyset \subseteq \mathbb{FP}^n$ is the unique (-1)-dim projective subspace of $\mathbb{FP}^n$.

Given linear space $V \leq \mathbb{F}^{n+1}$ define the "orthogonal complement"

$$V^\perp \subseteq \mathbb{F}^{n+1}$$

$$V^\perp = \{ \vec{x} \in \mathbb{F}^{n+1} : \vec{v} \cdot \vec{x} = 0 \ \forall \vec{v} \in V \}$$

"Rank - Nullity Theorem" of linear algebra

$$\Longrightarrow \quad \dim V + \dim V^\perp = n+1,$$

and it follows that $V^{\perp\perp} = V$.

It's a dimension argument!

Then we define the "projective dual" of a projective subspace:

$$\mathbb{P}(V)^\vee := \mathbb{P}(V^\perp)$$

and it follows that

$$\mathbb{P}(V)^{\vee\vee} = \mathbb{P}(V^\perp)^\vee$$

$$= \mathbb{P}(V^{\perp\perp}) = \mathbb{P}(V),$$

$$\begin{matrix} d\text{-dim proj} \\ \text{subsp. } \mathbb{FP}^n \end{matrix} \overset{\vee}{\longleftrightarrow} \begin{matrix} (n-1-d)\text{-dim proj} \\ \text{subsp. } \mathbb{FP}^n \end{matrix}$$

$$\text{e.g. } \underset{0\text{-dim}}{\text{points}} \longleftrightarrow \underset{(n-1)\text{-dim}}{\text{hyperplanes}}$$

A 0-dim proj subspace is just a point $\vec{a} \in \mathbb{FP}^n$. The proj dual is the hyp

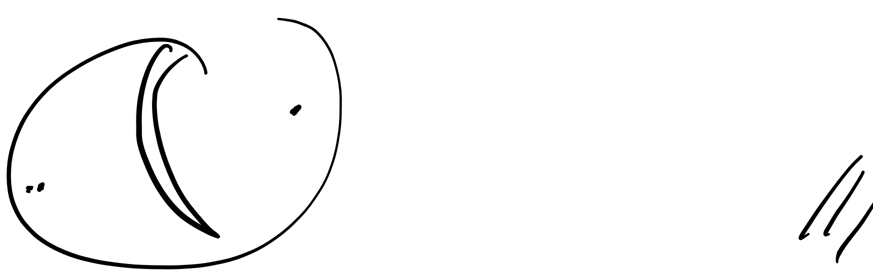
$$\{\vec{a}\}^\vee = H_{\vec{a}} : a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0.$$

It follows for indirect reasons that

$$\check{H}_{\vec{a}} = \{\vec{a}\},$$

which is difficult if you try to check directly.

e.g. in $\mathbb{RP}^2$ point-line duality are great circles and poles.



I claim that any d-dim proj subspace is projectively equivalent to the standard embedding of $\mathbb{RP}^d$:

$$\mathbb{RP}^d \subseteq \mathbb{RP}^n$$

$$\mathbb{RP}^d = \mathbb{P}(t_1\vec{e}_1 + \cdots + t_{d+1}\vec{e}_{d+1})$$

where $\vec{e}_1, \ldots, \vec{e}_{n+1} \in \mathbb{F}^{n+1}$ are the standard basis. Equivalently,

$$\mathbb{RP}^d = H_{d+2} \cap \cdots \cap H_{n+1}$$

where $H_i : x_i = 0$ are coordinate hyper-planes.

Proof: Let $\mathbb{P}(V) \subseteq \mathbb{FP}^n$ be d-dim. By definition, $V = t_1\vec{a}_1 + \cdots + t_{d+1}\vec{a}_{d+1}$ for some independent vectors

$$\vec{a}_1, \ldots, \vec{a}_{d+1} \in \mathbb{F}^{n+1}.$$

Let $A \in GL_{n+1}(\mathbb{F})$ be any invertible matrix with first $d+1$ columns equal to $\vec{a}_1, \dots, \vec{a}_{d+1}$. Then

$$A \cdot \mathbb{FP}^d = \mathbb{P}(V).$$

$$\mathbb{FP}^d = A^{-1} \cdot \mathbb{P}(V).$$

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Example: Any 1-dim proj subspace (line) $L \subseteq \mathbb{FP}^2$ has the form

$$L : s\vec{p} + t\vec{v}$$

where $\vec{p}, \vec{v}$ are two distinct points in $\mathbb{FP}^n$. Get a bijection

$$L \longleftrightarrow \mathbb{FP}^1$$

$$s\vec{p} + t\vec{v} \longleftrightarrow (s:t)$$

We can think

$\vec{p} + t\vec{v}$ = finite points of L

$\vec{v}$ = the infinite point of L

Use this to define the intersection multiplicity of a Line L & hypersurface V in projective space.

Say $L: t_1\vec{u}_1 + t_2\vec{u}_2$

$$V = V_F : F(\vec{x}) = 0$$

where F is homogeneous of deg d.

line $L = \underset{(n+1)\times 1}{U\vec{t}} = \underset{(n+1)\times 2}{\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}}$

Equation

$$\varphi(t_1, t_2) := F(U\vec{t}) = 0.$$

$\varphi(t_1, t_2) \in \mathbb{F}[t_1, t_2]$

homogeneous of degree d.

... continued next time.