Today: Sylvester's Determinant.

{This is how van der Waerden did algebraic geometry.}

Recall from the proof of Study's Lemma:

If $f, g \in \mathbb{F}[x_1, \dots, x_n]$ are coprime, then

$\exists\ \tilde{f}, \tilde{g} \in \mathbb{F}[x_1, \dots, x_n]$ & $h \in \mathbb{F}[\vec{x}_i']$

where $\vec{x}_i' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

such that

$$f\tilde{f} + g\tilde{g} = h.$$

This is some sort of weak version of Bézout's identity for UFDs.

Sylvester's Resultant Theorem:

Let R be UFD, so $R[x]$ also UFD.

Let $f, g \in R[x]$ have degrees d & e:

$$f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d$$
$$g(x) = b_0 x^e + \cdots + b_{e-1} x + b_e .$$

TFAE:

(1) f, g <u>not</u> coprime in $R[x]$

(2) $\exists\, \varphi, \gamma \in R[x]$ such that

- $\deg \varphi < \deg f$, $\deg \gamma < \deg g$,
- $f\gamma = g\varphi$.

(3) Sylvester's "resultant" is $\neq 0$:

$$\mathrm{Res}(f,g) = \det \begin{array}{c} \left(\begin{matrix} a_0 & \cdots & a_d & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_d \\ b_0 & \cdots & b_e & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_e \end{matrix}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} e \\ \left.\vphantom{\begin{matrix} b \\ b \\ b \end{matrix}}\right\} d \end{array}$$

[Square matrix of size $d+e$.]

Proof: (1) $\Leftrightarrow$ (2):

If $h \mid f$ & $h \mid g$ with h non-constant.

Hence $f = h\varphi$ & $g = h\gamma$ with

$\deg \varphi < \deg f$ & $\deg \gamma < \deg g$,
hence $f\gamma = h\varphi\gamma = g\varphi$. ///

Conversely, let $f\gamma = g\varphi$ for some $\deg \varphi < \deg f$ & $\deg \gamma < \deg g$, and suppose for contradiction f, g coprime. Since $f \mid g\varphi$ it follows from unique factorization that $f \mid \varphi$, contradicting the fact $\deg \varphi < \deg f$. ///

(2) $\Longleftrightarrow$ (3):
The desired φ, γ have the form

$$\varphi = u_0 x^{d-1} + \cdots + u_{d-2} x + u_{d-1},$$
$$\gamma = v_0 x^{e-1} + \cdots + v_{e-2} x + v_{e-1}.$$

Solve for the $d+e$ coefficients

$$u_0, \ldots, v_{e-1}.$$

The equation $f\gamma = g\varphi$ says

$$(a_0 x^d + \cdots + a_d)(v_0 x^{e-1} + \cdots + v_{e-1})$$

$$= (b_0 x^e + \cdots + b_e)(u_0 x^{d-1} + \cdots + u_{d-1})$$

Expand & compare x-coefficients:

$$\begin{aligned} a_0 v_0 &= b_0 u_0 \\ a_1 v_0 + a_0 v_1 &= b_1 u_0 + b_0 u_1 \\ &\ \vdots \\ a_d v_{e-1} &= b_e u_{d-1} \end{aligned}$$

This system has a solution $\Longleftrightarrow$ det of coeff matrix is nonzero:

$$\det \begin{pmatrix} a_0 & & & -b_0 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & a_0 & -b_e & & -b_0 \\ a_d & & \vdots & & \ddots & \vdots \\ & \ddots & & & & \\ & & a_d & & & -b_e \end{pmatrix} \neq 0.$$

Multiply b-columns by -1 & transpose.

Examples:

- A poly $f \in \mathbb{C}[x]$ has multiple root $\alpha \in \mathbb{C} \Leftrightarrow f, f'$ have common divisor $x - \alpha \in \mathbb{C}[x]$. If $f(x) = ax^2 + bx + c$, $f'(x) = 2ax + b$, then

$$\operatorname{Res}(f, f') = \det \begin{pmatrix} a & b & c \\ 2a & b & \\ & 2a & b \end{pmatrix}$$

$$= a(b^2 - 0) - 2a(b^2 - 2ac)$$

$$= -a(b^2 - 4ac)$$

Call $\operatorname{Res}(f, f')$ the "discriminant."

- Descartes: $f(x) = a_0 x^d + \cdots + a_d \in R[x]$ has root $\alpha \in R$ iff $f(x)$ & $g(x) = x - \alpha$ are <u>not</u> coprime. Indeed, one can show

$$\operatorname{Res}(f, x - \alpha) = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_d \\ 1 & -\alpha & & \\ & 1 & -\alpha & \\ & & \ddots & \ddots \\ & & 1 & -\alpha \end{pmatrix}$$

$$= (-1)^d (a_0 \alpha^d + \cdots + a_d)$$

$$= (-1)^d f(\alpha).$$

If $f(x) = \prod_{i=1}^{d} (x - \lambda_i)$ then this becomes

$$\mathrm{Res}(f, x-\alpha) = (-1)^d \prod (\alpha - \lambda_i)$$

$$= \prod (\lambda_i - \alpha).$$

- More generally, we will see that if $f(x) = \prod (x-\lambda_i)$ & $g(x) = \prod (x - \mu_i)$ then $\mathrm{Res}(f,g) = \prod_{i,j} (\lambda_i - \mu_j)$.

Resultant of Multivariate Polynomials:

Given $f, g \in R[\vec{x}]$ in n variables and x_i, expand

$$f(\vec{x}) = a_0(\vec{x}_i{}') x_i^d + \cdots + a_d(\vec{x}_i{}'),$$

$$g(\vec{x}) = b_0(\vec{x}_i{}') x_i^e + \cdots + b_e(\vec{x}_i{}').$$

Define the resultant with respect to the variable x_i:

$$\mathrm{Res}_{x_i}(f,g) = \det \begin{pmatrix} a_0 & \cdots & a_d & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_d \\ b_0 & \cdots & b_e & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_e \end{pmatrix}$$

$$\in R[\vec{x}_i{}'].$$

Some useful facts:

(a) $\mathrm{Res}_{x_i}(f,g) = (-1)^{de}\, \mathrm{Res}_{x_i}(g,f)$

(b) $\exists\ \tilde{f}, \tilde{g} \in R[\vec{x}]$ such that

$$\mathrm{Res}_{x_i}(f,g) = f\tilde{f} + g\tilde{g}.$$

(c) If f & g are homogeneous of degrees d & e, then a_k & b_k are homogeneous of degree k. Then one of the following holds:

- $\mathrm{Res}_{x_i}(f,g) = 0$.

- $\mathrm{Res}_{x_i}(f,g)$ is <u>homogeneous</u> of degree $de = \deg(f)\cdot\deg(g)$.

(d) If $f, g \in R[x]$ split
$f = \prod(x-\lambda_i)$ & $g = \prod(x-\mu_j)$
then $\mathrm{Res}(f,g) = \prod_{i,j}(\lambda_i - \mu_j)$.

(e) For any f_1, f_2, g we have

$$\mathrm{Res}(f_1 f_2, g) = \mathrm{Res}(f_1, g)\,\mathrm{Res}(f_2, g).$$

<u>Proof</u>: (a): Cyclic permutation of the rows multiplies det by $(-1)^{d+e-1}$, since it can be achieved by sequence of $d+e-1$ adjacent row swaps. To put b-rows at the top we need to do this e times:

$$(-1)^{(d+e-1)e} = (-1)^{de}\cancel{(-1)^{e(e-1)}} = (-1)^{de} \checkmark$$

(b) Replace 1st column by

$$\sum_{k=1}^{d+e} x_i^{k-1} \text{ (} k\text{th column)},$$

which does <u>not</u> change the determinant.

The new 1st column is

$$\left(f(\vec{x}), x_i f(\vec{x}), \dots, x_i^{d-1} f(\vec{x}), \right.$$
$$\left. g(\vec{x}), x_i g(\vec{x}), \dots, x_i^{e-1} g(\vec{x}) \right)$$

Expand determinant along 1st column:

$$\mathrm{Res}_{x_i}(f,g) = f\tilde{f} + g\tilde{g} \qquad \checkmark$$

(c) If $a_k, b_k \in \mathbb{F}[\vec{x}_i']$ are hom. of degree k then

$$a_k(\lambda \vec{x}_i') = \lambda^k a_k(\vec{x}_i')$$
$$b_k(\lambda \vec{x}_i') = \lambda^k b_k(\vec{x}_i')$$

Thus:

$$\mathrm{Res}_{x_i}(f,g)(\lambda \vec{x}_i') = \begin{pmatrix} a_0 & \lambda a_1 & \cdots & \lambda^d a_d & & \\ & \ddots & & & \ddots & \\ & & a_0 & \cdots & \cdots & \lambda^d a_d \\ b_0 & \cdots & \lambda^e b_e & & & \\ & \ddots & & & \ddots & \\ & & b_0 & \cdots & \cdots & \lambda^e b_e \end{pmatrix}$$

TRICK: Multiply first e rows by $\lambda, \lambda^2, \dots, \lambda^e$ & multiply final d rows by $\lambda, \lambda^2, \dots, \lambda^d$. Net result is to multiply k^{th} column of the original Sylvester matrix by λ^k. Hence

$$\lambda \cdots \lambda^d \lambda \cdots \lambda^e \, \mathrm{Res}(\lambda \vec{x}_i') = \lambda \cdots \lambda^{d+e} \, \mathrm{Res}(\vec{x}_i')$$

$$\lambda^{d(d+1)/2} \lambda^{e(e+1)/2} \mathrm{Res}(\lambda \vec{x}_i') = \lambda^{\frac{(d+e)(d+e+1)}{2}} \mathrm{Res}(\vec{x}_i')$$

$$\mathrm{Res}(\lambda \vec{x}_i') = \lambda^{de} \, \mathrm{Res}(\vec{x}_i')$$

MAGIC!