Today: Projective Tangent Spaces.

This will finish the story of Taylor series, from the 19th century point of view.

Given line L & hypersurface V_F in $\mathbb{FP}^n$ we have defined the intersection multiplicity $[L \cdot V_F]_p \in \mathbb{N}$ and shown it is invariant under:

- automorphisms of $\mathbb{FP}^n$
- automorphisms of $L \cong \mathbb{FP}^1$

We say L & V_F are tangent if

$$[L \cdot V_F]_{\vec{p}} \geq 2.$$

We will use this define/compute the projective tangent space.

To do this, let $F(\vec{x}) \in \mathbb{F}[x_1, \dots, x_{n+1}]$ be homogeneous & parametrize the line as $L: \vec{p} + t\vec{q}$ for distinct $\vec{p}, \vec{q} \in \mathbb{FP}^n$.

For a specific representation

$$\vec{p} = (p_1, p_2, \dots, p_{n+1})$$

we compute the Taylor expansion of F near $\vec{x} = \vec{p}$:

$$F(\vec{x}) = F(\vec{p}) + (\nabla F)_{\vec{p}} (\vec{x} - \vec{p})$$
$$+ \frac{1}{2} (\vec{x} - \vec{p})^T (HF)_{\vec{p}} (\vec{x} - \vec{p}) + \cdots$$

Choose specific representation

$$\vec{q} = (q_1, \dots, q_{n+1})$$

and substitute $L: \vec{p} + t\vec{q}$ into F:

$$\Phi(t) := F(\vec{p} + t\vec{q})$$
$$= F(\vec{p}) + t (\nabla F)_{\vec{p}} \vec{q} + \frac{t^2}{2} \vec{q}^T (HF)_{\vec{p}} \vec{q} + \cdots$$

By definition, $[L \cdot V_F]_{\vec{p}}$ is the multiplicity of $t=0$ as a root of $\Phi(t)$. So L & V_F are tangent iff

- $F(\vec{p})=0$
- $(\nabla F)_{\vec{p}}\,\vec{q}=0$.

If $(\nabla F)_{\vec{p}} \neq \vec{0}$, the second equation defines a projective hyperplane $T_{\vec{p}}V_F = H_{(\nabla F)_{\vec{p}}}$ called the projective tangent space to V_F at $\vec{p}$.

If $(\nabla F)_{\vec{p}} = \vec{0}$ then every line through $\vec{p}$ is tangent to V_F, so we define

$$T_{\vec{p}}V_F = \mathbb{F}\mathbb{P}^n.$$

We say $\vec{p} \in V_F$ is smooth (regular) when $\dim T_{\vec{p}}V_F = n-1$ and singular when $\dim T_{\vec{p}}V_F = n$, as projective subspaces.

Next, want to understand affine vs. projective tangent spaces. The key fact connecting them is

Euler's Homogeneous Function Theorem:
Let $F(\vec{x}) \in \mathbb{F}[x_1, \dots, x_{n+1}]$ over field $\mathbb{F}$ and consider 3 conditions:

(H1) $F(\vec{x}) = F^{(d)}(\vec{x})$

(H2) $F \neq 0$ & $F(\lambda \vec{x}) = \lambda^d F(\vec{x}) \ \forall \lambda \neq 0$.

(H3) $d \cdot F(\vec{x}) = (\nabla F)_{\vec{x}} \vec{x}$

$$= x_1 F_{x_1}(\vec{x}) + \dots + x_{n+1} F_{x_{n+1}}(\vec{x}).$$

Then (H1) ⇒ (H2) & (H1) ⇒ (H3) always.
(H2) ⇒ (H1) if $\mathbb{F}$ infinite
(H3) ⇒ (H1) if char$(\mathbb{F}) = 0$. ///

We already discussed H1 & H2.

<u>Proof</u>: (H1) ⇒ (H3):

Assume $F(\vec{x}) = F^{(d)}(\vec{x}) = \sum a_I \vec{x}^I$

where each $I = (i_1, \dots, i_{n+1})$ satisfies

$\sum I = i_1 + \dots + i_{n+1} = d$. Note that for

each x_k and each monomial $\vec{x}^I$ we

have $x_k D_{x_k} \vec{x}^I = i_k \vec{x}^I$, hence

$$
\begin{aligned}
(\nabla F)_{\vec{x}} \vec{x} &= \sum x_k D_{x_k} F \\
&= \sum_I a_I \sum_k x_k D_{x_k} \vec{x}^I \\
&= \sum_I a_I \sum_k i_k \vec{x}^I \\
&= \sum_I a_I (\underbrace{i_1 + \dots + i_{n+1}}_{d}) \vec{x}^I \\
&= d \sum a_I \vec{x}^I \\
&= d \cdot F(\vec{x}).
\end{aligned}
$$

$(H3) \Rightarrow (H1)$: Assume $d \cdot F = (\nabla F)_{\vec{x}} \vec{x}$.

Assume $\operatorname{char}(\mathbb{F}) = 0$.

Let $F(\vec{x}) = \sum F^{(k)}(\vec{x})$ be

homogeneous filtration.

Since $(\nabla F)_{\vec{x}}\,\vec{x}$ is linear in F we have

$$d\,F = (\nabla F)_{\vec{x}}\,\vec{x}$$
$$= \sum_k (\nabla F^{(k)})_{\vec{x}}\,\vec{x}$$
$$= \sum_k k\cdot F^{(k)} \quad \text{since } (H1) \Rightarrow (H3).$$

Now let y be another variable and substitute $\vec{x} \mapsto y\vec{x}$. Then since $(H1) \Rightarrow (H2)$ we have

$$d\cdot F(y\vec{x}) = \sum k\cdot F^{(k)}(y\vec{x})$$
$$d\sum_k F^{(k)}(y\vec{x}) = \sum k\cdot F^{(k)}(y\vec{x})$$
$$\sum_k d\,y^k F^{(k)}(\vec{x}) = \sum k\,y^k F^{(k)}(\vec{x})$$

This is an identity of polynomials in the ring $\mathbb{F}[\vec{x}][y]$, hence the coefficient of y^k on each side is the same:

$$d\,F^{(k)}(\vec{x}) = k\,F^{(k)}(\vec{x})$$
$$(d-k)\,F^{(k)}(\vec{x}) = 0.$$

Since $\mathrm{char}(\mathbb{F}) = 0$ then $d \neq k$ in $\mathbb{Z}$ implies $d - k \neq 0$ in $\mathbb{F}$, hence

$$F^{(k)}(\vec{x}) = 0.$$

We conclude that $F(\vec{x}) = F^{(d)}(\vec{x})$.

We use this lemma to relate affine & projective tangent spaces.

Theorem: Let $\vec{p} \in V_F \subseteq \mathbb{F}\mathbb{P}^n$ be a point on a projective hypersurface, let $T_{\vec{p}} V_F$ be projective tangent space. If $\vec{p} \notin H_i$, let f be ith de-homog. of F. Then I claim that affine tangent space $T_{\vec{p}} V_f \subseteq U_i \subseteq \mathbb{F}\mathbb{P}^n$ is the dehomog. of $T_{\vec{p}} V_F$ in U_i.

Conversely, if V_F does not contain H_i (ie. if $x_i \nmid F$) then $T_{\vec{p}} V_F$ is the ith homogenization of $T_{\vec{p}} V_f$.

Geometric Meaning: The tangent space at a point is determined by any local neighborhood.

Proof: The proj. tangent space is defined by equation

$$(\nabla F)_{\vec{p}} \, \vec{x} = 0 \qquad (*)$$

$$x_1 F_{x_1}(\vec{p}) + \cdots + x_{n+1} F_{x_{n+1}}(\vec{p}) = 0$$

Let $\vec{p} = (p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_{n+1}) \in U_i$.

If $k \neq i$ we observe that D_{x_k} commutes with substituting $x_i = 1$.

The affine tangent space $T_p V_f \subseteq U_i$ is defined by equation

$$f_{x_1}(\vec{p})(x_1 - p_1) + \cdots + f_{x_{i-1}}(\vec{p})(x_{i-1} - p_{i-1})$$

$$+ f_{x_{i+1}}(\vec{p})(x_{i+1} - p_{i+1}) + \cdots + f_{n+1}(\vec{p})(x_{n+1} - p_{n+1}) = 0.$$

Now homogenize in the ith place:

$$F_{x_1}(\vec{p})(x_1 - p_1 x_i) + \cdots + F_{x_{i-1}}(\vec{p})(x_{i-1} - p_{i-1} x_i) \quad (**)$$

$$+ F_{x_{i+1}}(\vec{p})(x_{i+1} - p_{i+1} x_i) + \cdots + F_{n+1}(\vec{p})(x_{n+1} - p_{n+1} x_i) = 0$$

Is this the same as $(*)$?

Yes because of Euler's formula:

$$d \cdot F(\vec{p}) = (\nabla F)_{\vec{p}} \, \vec{p}$$

$$0 = p_1 F_{x_1}(\vec{p}) + \cdots + p_{i-1} F_{x_{i-1}}(\vec{p})$$

$$+ 1 F_{x_i}(\vec{p}) + p_{i+1} F_{x_{i+1}}(\vec{p}) + \cdots + p_{n+1} F_{x_{n+1}}(\vec{p}).$$

$$\Rightarrow F_{x_i}(\vec{p}) = -p_1 F_{x_1}(\vec{p}) - \cdots - p_{n+1} F_{x_{n+1}}(\vec{p}).$$

Substitute into $(**)$ to get $(*)$.

Next time: Abstract Algebra!