Zariski Topology Continued:

$$I : (\text{subsets } \mathbb{F}^n) \rightleftarrows (\text{ideals } \mathbb{F}[\vec{x}]) : V$$

$$V(I) := \{\vec{p} \in \mathbb{F}^n : f(\vec{p}) = 0 \ \forall f \in I\}$$

$$I(S) := \{f \in \mathbb{F}[\vec{x}] : f(\vec{p}) = 0 \ \forall \vec{p} \in S\}.$$

Observe: $I(S)$ is an ideal.

key property:

$$S \subseteq V(I) \iff \forall \vec{p}\, S,\ \forall f \in I,\ f(\vec{p}) = 0$$

$$\iff \forall f \in I,\ \forall \vec{p} \in S,\ f(\vec{p}) = 0$$

$$\iff I \subseteq I(S)$$

Hence I, V is a Galois connection, so we get lots of results for free!

Note: subsets of $\mathbb{F}^n$ is a lattice:

$$\vee = \cup \ \& \ \wedge = \cap.$$

ideals of $\mathbb{F}[\vec{x}]$ is a lattice:

$$\vee = + \ \& \ \wedge = \cap.$$

So for any collections of sets $\{S_i \subseteq \mathbb{F}^n\}$ and ideals $\{I_i \subseteq \mathbb{F}[\vec{x}]\}$ we have

$$\sum_i I(S_i) \subseteq I(\cup_i S_i)$$

$$\cup_i V(I_i) \subseteq V(\cap_i I_i)$$

$$\cap_i I(S_i) = I(\cup_i S_i)$$

$$\cap_i V(I_i) = V(\textstyle\sum_i I_i)$$

We also have an order-reversing bijection between "closed elements":

$$I : (\text{closed sets}) \xleftrightarrow{\sim} (\text{closed ideals}) : V$$

S closed $\iff$ $V(I(S)) = S$

I closed $\iff$ $I(V(I)) = I$.

Which S & I satisfy this?

Definition: Closure VI is called Zariski closure, closed sets are called varieties. In fact, this defines a topology because we also have

$$V(I_1 \cap I_2) = V(I_1) \cup V(I_2).$$

One direction is purely formal.
Other direction: last time.

Hilbert Basis Theorem:

Varieties = finite intersections of hypersurfaces.

<u>Proof</u>: Variety has the form $V(I)$ for some ideal. HBT: I has form

$$I = f_1 \mathbb{F}[\vec{x}] + \cdots + f_m \mathbb{F}[\vec{x}].$$

Hence

$$V(I) = V_{f_1} \cap \cdots \cap V_{f_m}. \quad ///$$

Remark: # hypersurfaces required might be larger than you expect.

i.e. if "dim V" d, might require $\gg n-d$ hypersurfaces.

Strong Nullstellensatz:

If $\mathbb{F}$ algebraically closed, then for any ideal $I \subseteq \mathbb{F}[\vec{x}]$ have

$$I(V(I)) = \sqrt{I}$$
$$= \{g : g^r \in I \text{ for some } r\}.$$

Unions of varieties are much easier.

Theorem (Unique Factorization):

i) Every variety has a unique minimal decomposition into irreducibles:

$$V = V_1 \cup \cdots \cup V_k,$$

"minimal": $V_i \nsubseteq V_j$ for $i \neq j$.

ii) Let V, I be variety, rad ideal pair. Then

$$V \text{ irreducible} \iff I \text{ prime}.$$

[Remark: Primes are radical.

$$g \notin P \Rightarrow g^r \notin P. \quad]$$

iii) Every rad ideal I is a unique minimal intersection of primes:

$$I = P_1 \cap \cdots \cap P_k.$$

"minimal": $P_i \not\subseteq P_j$ for $i \neq j$.

<u>Proof</u> (i): Existence. If V cannot be expressed as union of irreducibles, get infinite <u>strictly</u> descending chain of varieties, which maps to infinite <u>strictly</u> ascending chain of ideals. $\lightning$

[I is injective on varieties.]

Uniqueness: Suppose

$$V_1 \cup \cdots \cup V_k = V_1' \cup \cdots \cup V_\ell'$$

$V_i \not\subseteq V_j$ & $V_i' \not\subseteq V_j'$ for $i \neq j$.

For any i,

$$V_i \subseteq V_1' \cup \cdots \cup V_\ell'$$

$$V_i = V_i \cap (V_1' \cup \cdots \cup V_\ell')$$

$$V_i = (V_i \cap V_1') \cup \cdots \cup (V_i \cap V_\ell')$$

$\Rightarrow V_i = V_i \cap V_j'$ some j (V_i is irr.)

$$V_j' \subseteq V_i.$$

Symmetry: $V_k \subseteq V_j'$ some k.

$$V_k \subseteq V_j' \subseteq V_i$$

$\Rightarrow V_k \subseteq V_i \Rightarrow k = i$

$\Rightarrow V_i = V_j' = V_i.$

Conclusion: Each summand on left = some summand on right & vice versa.

Done.

iii) Not much to do.

ii) Easy direction (arbitrary $\mathbb{F}$):

I not prime $\Rightarrow$ V reducible.

Suppose I not prime.

$\exists\, f_1, f_2 \notin I,\ f_1 f_2 \in I.$

$V_1 := V(I + f_1\mathbb{F}[\vec{x}])$

$V_2 := V(I + f_2\mathbb{F}[\vec{x}])$.

Claim: $V = V_1 \cup V_2$ & $V_1, V_2 \neq V$.

Since $I \subseteq I + (f_1)$, $I \subseteq I + (f_2)$, get

$V_1 \subseteq V$ & $V_2 \subseteq V$, hence $V_1 \cup V_2 \subseteq V$.

To show $V_1, V_2 \neq V$:

Since $f_1, f_2 \notin I$ $\exists\, \vec{p}_1, \vec{p}_2 \in V$ with

$f_1(\vec{p}_1) \neq 0$ & $f_2(\vec{p}_2) \neq 0$,

hence $\vec{p}_1 \notin V_1$ & $\vec{p}_2 \notin V_2$.

To show $V \subseteq V_1 \cup V_2$. Let $\vec{p} \in V$.

Since $f_1 f_2 \in I$, $f_1 f_2(\vec{p}) = 0$

$f_1(\vec{p}) f_2(\vec{p}) = 0.$

$\Rightarrow f_1(\vec{p}) = 0$ or $f_2(\vec{p}) = 0$

$\vec{p} \in V_1$ $\qquad$ $\vec{p} \in V_2$.

Hard Direction (uses NSS):

V reducible $\Rightarrow$ I not prime.

Let $V = V_1 \cup V_2$, $V_1, V_2 \neq V$.

Let $I_1 = I(V_1)$ & $I_2 = I(V_2)$.

Will show $\exists f_1, f_2 \notin I$, $f_1 f_2 \in I$.

NSS: I injective on radical ideals.

$V_1, V_2 \neq V \Rightarrow I_1, I_2 \neq I$.

$\exists f_1 \in I \setminus I_1$ & $f_2 \in I \setminus I_2$.

To show $f_1 f_2 \in I$, note $\forall \vec{p} \in V$

have $\vec{p} \in V_1$ or $\vec{p} \in V_2$.

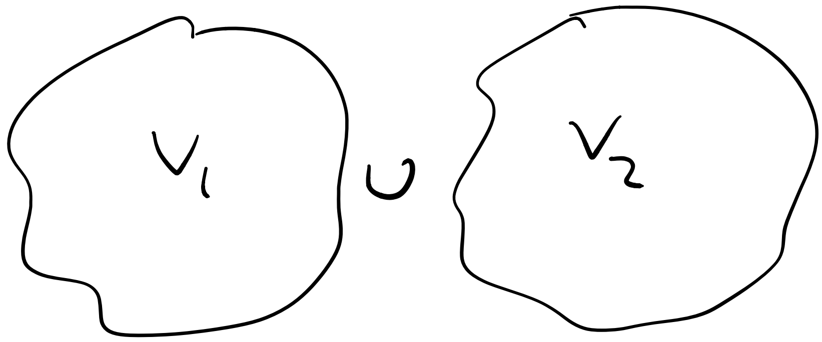
$f_1(\vec{p}) = 0$ $\qquad$ $f_2(\vec{p}) = 0$

$\Rightarrow f_1 f_2(\vec{p}) = f_1(\vec{p}) f_2(\vec{p}) = 0$. QED.

Intuition:

Irreducible = connected

Polynomial = analytic function

$V =$ 

Disconnected.

$\exists$ continuous functions

$$f_1, f_2 : V \to \mathbb{F}$$

where $f_1 \neq 0$ on V_1 $\quad f_2 = 0$ on V_1

$\qquad\quad f_1 = 0$ on V_2 $\quad f_2 \neq 0$ on V_2

Then $f_1, f_2 \notin I(V)$

But $f_1 f_2 \in I(V)$

because $f_1 f_2$ is zero on V_1 & V_2.