Zariski Topology Continued:

Abstract Galois Connection:

$(P, \leq), (Q, \leq)$ posets

$$* : P \rightleftarrows Q : *$$

$$p \leq q^* \iff q \leq p^*.$$

(a) $p \leq p^{**}$

(b) $p_1 \leq p_2 \Rightarrow p_2^* \leq p_1^*$

(c) $p^* = p^{***}$

(d) Say p <u>closed</u> if $p = p^{**}$.

$Q^* = \{q^* : q \in Q\} = \{p : p = p^{**}\} \subseteq P$.

(e) Order-Reversing bijection of closed elements:

$$* : Q^* \overset{\sim}{\longleftrightarrow} P^* : *$$

(f) If P, Q have least upper bounds $\vee$ & greatest lower bounds $\wedge$, then

$$\bigwedge_i p_i^* = (\bigvee_i p_i)^*$$

$$\bigvee_i p_i^* \leq (\bigwedge_i p_i)^* \quad (\neq \text{ in general})$$

Follows that g.l.b. of closed elements is closed.

<u>Proof</u>: (d): $p = p^{**}$ closed then

$p = (p^*)^* \in Q^*$. Conversely,

if $q^* \in Q^*$ then $(q^*)^{**} = q^*$ ✓

(f): By def: $p_j \leq \bigvee_i p_i \quad \forall j$.

$\Rightarrow (\bigvee_i p_i)^* \leq p_j^* \quad \forall j$

a lower bound of set $\{p_j\}$.

$\Rightarrow (\bigvee_i p_i)^* \leq \bigwedge_i p_i^*$.

Conversely: By def, $(\bigwedge_j p_j^*) \leq (p_i)^*$

for all i. By def of Galois connection:

$(p_i) \leq (\bigwedge_j p_j^*)^* \; \forall i$.

upper bound of set $\{p_i\}$.

$\Rightarrow (\vee_i p_i) \le (\wedge_j p_j^*)^*$

$\Rightarrow \wedge_j p_j^* \le (\vee_i p_i)^*$ ✓

Galois definition.

QED.

Galois Closure Space (Kuratowski, 1921).

Let X be set, $cl : 2^X \to 2^X$ function.

Called a "Galois closure" if

(T1) $S \subseteq cl(S)$

(T2) $S \subseteq T \Rightarrow cl(S) \subseteq cl(T)$

(T3) $cl(S) = cl(cl(S))$

(T4) intersection of closed sets is closed.

If $* : 2^X \rightleftarrows 2^Y : *$ is Galois conn.

then $** : 2^X \to 2^X$ & $** : 2^Y \to 2^Y$

are Galois closures.

(T1): (a).

(T2): (b) twice

$S \subseteq T \rightsquigarrow T^* \subseteq S^* \rightsquigarrow S^{**} \subseteq T^{**}$.

(T3): (c) $S^* = S^{***} \rightsquigarrow S^{**} = (S^{**})^{**}$.

(T4): (F).

Not yet a modern topology.
Require two more properties:

(T5): $\emptyset$ is closed.

(T6): finite union of closed is closed.

Not automatic. But they do hold for certain Galois connections, e.g., Zariski.

Zariski Topology:
$\mathbb{F}$ field, $\mathbb{F}[\vec{x}] = \mathbb{F}[x_1, \dots, x_n]$. Define

$$I : (\text{subsets } \mathbb{F}^n) \rightleftarrows (\text{ideals } \mathbb{F}[\vec{x}]) : V$$

$$V(I) = \{ \vec{p} : f(\vec{p}) = 0 \ \forall f \in I \}$$
$$I(S) = \{ f : f(\vec{p}) = 0 \ \forall \vec{p} \in S \}.$$

This is an ideal. $f, g \in I(S)$, $h \in \mathbb{F}[\vec{x}]$.
For all $\vec{p} \in S$, $(f+gh)(\vec{p}) = f(\vec{p}) + g(\vec{p})h(\vec{p})$
$= 0 + 0\, h(\vec{p}) = 0$, hence $f+gh \in I(S)$.

Claim: Galois connection.

$S \subseteq V(I) \iff \forall \vec{p} \in S, \forall f \in I, f(\vec{p}) = 0.$

$\iff \forall f \in I, \forall \vec{p} \in S, f(\vec{p}) = 0.$

$\iff I \subseteq I(S).$

Call $VI : 2^{\mathbb{F}^n} \to 2^{\mathbb{F}^n}$ Zariski closure.

Satisfies T1 — T4. But also satisfies T5, T6.

(T5): $\emptyset = V(\mathbb{F}[\vec{x}])$ is closed.

(T6): Finite union of closed is closed.

We will show

$$V(I_1) \cup V(I_2) = V(I_1 \cap I_2).$$

We have $\subseteq$ from Galois properties.

Conversely, suppose $\vec{p} \notin V(I_1) \cup V(I_2)$.

We will show $\vec{p} \notin V(I_1 \cap I_2)$.

Well since $\vec{p} \notin V(I_1), V(I_2)$, $\exists$

$f_1 \in I_1, f_2 \in I_2$, $f_1(\vec{p}) \neq 0, f_2(\vec{p}) \neq 0.$

But then $f_1 f_2 \in I_1 \cap I_2$ satisfies

$(f_1 f_2)(\vec{p}) = f_1(\vec{p}) f_2(\vec{p}) \neq 0,$

hence $\vec{p} \in V(I_1 \cap I_2)$. ///

So Zariski closure defines a legit topology on $\mathbb{F}^n$. Closed sets are called "varieties." Observe from HBT variety is finite intersection of hypersurfaces:

Every variety is $V(I)$ for some I.

HBT $\Rightarrow$ $I = f_1 \mathbb{F}[\vec{x}] + \cdots + f_m \mathbb{F}[\vec{x}]$

$\Rightarrow V(I) = V_{f_1} \cap \cdots \cap V_{f_m}$ ✓

Also have order-reversing bijection:

$$\underline{I} : (\text{varieties}) \xleftrightarrow{\sim} (\text{closed ideals}) : V.$$

What are the <u>closed</u> ideals?

Let $\mathbb{F}$ be algebraically closed.

Then (Strong Nullstellensatz):

$$\underline{I}(V(I)) = \sqrt{I} = \{ g : g^r \in I \text{ some } r \},$$

The "radical closure" of I.

Proof: $g \in \sqrt{I}$ then $g^r \in I$

$\Longrightarrow$ for all $\vec{p} \in V(I)$, $g^r(\vec{p}) = 0$

hence $g(\vec{p}) = 0$, hence $g \in I(V(I))$ ✓

Other direction is literally NSS. ///

Corollary: $\sqrt{I}$ is an ideal. (But there are much easier ways to prove this!)

Ideals $I = \sqrt{I}$ are called <u>radical</u>.

I: (varieties) $\overset{\sim}{\longleftrightarrow}$ (radical ideals): V

To conclude: Recall from Study's Lemma.

Hypersurface = unique union of irr. hypersurfaces.

Generalize to <u>lower</u> dimensions.

(i): Let V, I be variety, rad ideal pair.

V irreducible $\Longleftrightarrow$ I prime.

(ii) Unique min irr decomposition:

$V = V_1 \cup V_2 \cup \cdots \cup V_k$.

"minimal": $V_i \not\subseteq V_j \ \forall i \neq j$.

(iii) Radical ideal I contained in finitely many minimal primes $P_i \supseteq I$, and $I = P_1 \cap P_2 \cap \cdots \cap P_k$.

[Again: This is true in general ring, just not for finitely many.]

Also know: Variety is finite intersection of hypersurfaces, but this is much more complicated!