Preview of next semester (MTH 787).

Theory of one-dimensional varieties.

Bézout's Theorem:

Given $f(x,y)$ & $g(x,y) \in \mathbb{C}[x,y]$ of degrees d & e, find the intersection

$$C_f \cap C_g.$$

Bézout: If f & g coprime then

$$\# C_f \cap C_g \leq de.$$

And equal if we count with multiplicity & in $\mathbb{CP}^2$.

Proof (sketch): Homogenize:

$$\underset{d}{F(x,y,z)}, \underset{e}{G(x,y,z)} \in \mathbb{C}[x,y,z]$$

Note: $(a,b,c) \in C_F \cap C_G \iff$

$$F(x,y,c), G(x,y,c) \in \mathbb{C}[x,y]$$

have a common root $xb - ya$.

This happens $\iff \mathrm{Res}_z(F, G) = 0$.
We showed that

$$\mathrm{Res}_z(F, G) = \mathbb{C}[x, y]$$

is homogeneous of degree de,
hence it splits over $\mathbb{C}$:

$$\mathrm{Res}_z(F, G) = \prod_i (x b_i - y a_i)^{m_i}.$$

Say that m_i is the multiplicity of corresponding point of intersection

$$\Longrightarrow \quad \sum m_i = \deg \mathrm{Res}(F, G) = de. \quad ///$$

Corollaries:

- No disjoint curves in $\mathbb{CP}^2$.
- $< \infty$ singular points
$$C_f \cap C_{\frac{df}{dx}} \cap C_{\frac{df}{dy}}.$$
- $< \infty$ inflection points
$$C_f \cap C_{\det(H_f)}. \quad ///$$

Lots of interesting <u>enumerative</u> questions.

Plücker's formulas.

Intersection multiplicity turns out to be a projective invariant.

Modern View: Local Rings.

To irreducible curve C_F, we associate a ring $\mathbb{C}[C] = \mathbb{C}[x,y,z]/(F)$.

Since F is irreducible, (F) is prime, hence $\mathbb{C}[C]$ is a domain. Define the "field of rational functions"

$$\mathbb{C}(C) = \operatorname{Frac} \mathbb{C}[x,y,z]/(F).$$

Would like to think of these as functions $C \longrightarrow \mathbb{C}$, but this is not quite true.

Given $\vec{p} \in C$ define

$$\mathcal{O}_{\vec{p}}(C) = \{ f \in \mathbb{C}(C) : \exists \text{ expression}$$

$f = a/b$ such that $b(\vec{p}) \neq 0 \}$

= rational functions defined at $\vec{p}$.

Consider the maximal ideal

$$m_{\vec{p}} \subseteq \mathcal{O}_{\vec{p}}$$

$$m_{\vec{p}} = \{ f \in \mathcal{O}_{\vec{p}} : f(\vec{p}) = 0 \}$$

This is the unique maximal ideal, so $\mathcal{O}_{\vec{p}}$ is called a "local ring" (one maximal ideal = one point)

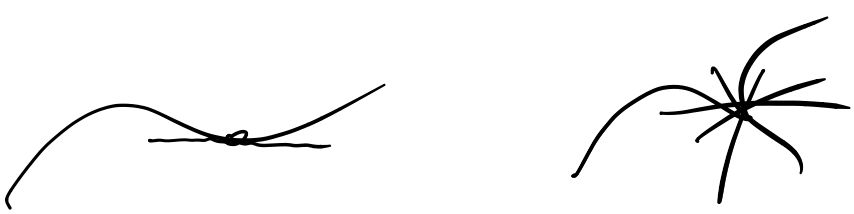
Zariski tangent space:

$$(T_{\vec{p}} C_F)^* \approx \frac{M_{\vec{p}}}{M_{\vec{p}}^2 + (F)} \approx \frac{m_{\vec{p}}}{m_{\vec{p}}^2}$$

Point $\vec{p} \in C$ is smooth

$$\iff \dim_{\mathbb{C}}(m_p / m_p^2) = 1$$

is singular $\iff \dim_{\mathbb{C}}(m_p/m_p^2) = 2$.



Algebraically:

$\vec{p} \in C$ smooth $\Longleftrightarrow$ $\mathcal{O}_p$ is DVR
"discrete valuation ring"

Let $\vec{p} \in C_F \cap C_G$. Define the intersection multiplicity:

$$[C_F \cdot C_G]_p = \dim_{\mathbb{C}} \mathcal{O}_p(\mathbb{C}^2) / (F, G)$$

Properties:
If $V(I) = \{p_1, p_2, \ldots, p_k\} \subseteq \mathbb{C}^2$ consists of finitely many points, let

$$\mathcal{O}_i = \mathcal{O}_{p_i}(\mathbb{C}^2)$$

Then we have an isomorphism:

$$\mathbb{C}[x,y]/I = \prod_i \mathcal{O}_i / I\mathcal{O}_i$$

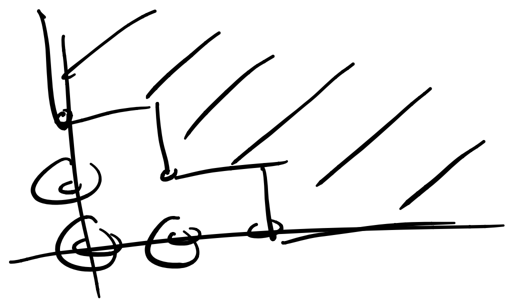
Corollary:

$$\sum_i [C_F \cdot C_G]_{p_i} = \dim \mathbb{C}[x,y]/(F,G).$$
$$= de.$$

Example: $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$

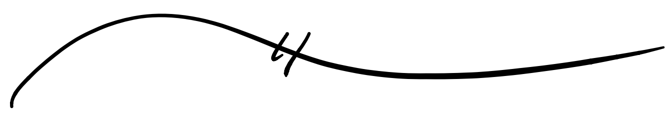
$V(m_p) \cup V(m_q) = V(m_p \cap m_q)$

$\mathbb{C}[x,y] / (x^2, xy, y^2)$



dim = 3

basis $1, x, y$.



Complex Analysis:

Given $f(x,y) \in \mathbb{C}[x,y]$ irreducible, the curve $C_f \subseteq \mathbb{C}^2$ is

- compact in $\mathbb{CP}^2$
- orientable
- connected.

Orientable: If $\alpha \in \mathbb{C}^2 = \mathbb{R}^4$ is tangent to curve at $\vec{p}$, then $i\alpha \in \mathbb{C}^2 = \mathbb{R}^4$ is also tangent.

Basis $\alpha, i\alpha$ of tangent space to C changes smoothly.

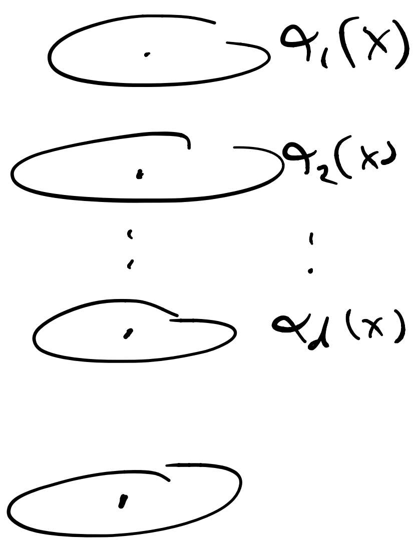
Connected: Let $f(x,y) = y^d + $ lower terms.

For each $\alpha \in \mathbb{C}$ define $f_\alpha(y) = f(\alpha, y)$ in $\mathbb{C}[y]$. Ramification points

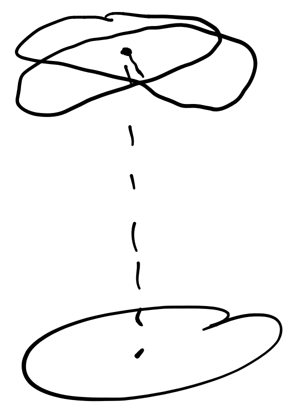
$$S = \{\alpha \in \mathbb{C} : f_\alpha(y) \text{ has repeated root}\}$$

$$\# S \leq d(d-1) < \infty.$$

Above each $x \in \mathbb{C} \setminus S$, $\exists$ exactly d roots of $f_x(y)$: $\alpha_1(x), \ldots, \alpha_d(x)$.



$x \in \mathbb{C} \setminus S$



$x \in S$

Ramified covering of degree d.

Over any $x \in \mathbb{C} \setminus S$ we have

$$f(x,y) = \prod_{i=1}^{d} (y - \alpha_i(x))$$

Follows: $e_k(x) = \prod_{i_1 < \cdots < i_k} \alpha_{i_1}(x) \cdots \alpha_{i_k}(x) \in \mathbb{C}[x]$.

Connected? Suppose not; $C = C_1 \amalg C_2$

degrees $d_1 + d_2 = d$.

$$f(x,y) = \prod_{i=1}^{d_1} (y - \beta_i(x)) \prod_{i=1}^{d_2} (y - \gamma_i(x))$$

are these polynomials?

Functions $f_k = \prod_{i_1 < \cdots < i_k} \beta_{i_1}(x) \cdots \beta_{i_k}(x)$

are holomorphic and grow like a polynomial, hence they are polynomial.

///

Note: Galois Theory!

Topology: Riemann-Hurwitz

$\chi_C = 2 - 2g$

$g = 3$

Let $\varphi : M \to N$ degree d.

$$\chi_M = d(\chi_N - k) + \sum e_i$$

where e_i is number of preimages over ramification point m_i.

Corollaries:

- any map $C \to \mathbb{P}^1$ with $\chi_C \geq 1$ is ramified.
- Genus of a smooth curve of deg d:

$$g = \frac{(d-1)(d-2)}{2}.$$

Number Theory:

Covering Spaces $\approx$ Galois Theory
Riemann Surfaces $\approx$ Number Theory

$\mathbb{C}[z]$ $\quad\quad$ $\mathbb{Z}$

PIDs

$$\begin{tikzcd} C \arrow[d, "d"'] & \mathbb{C}(C) & \supseteq & \mathbb{C}[C] \\ \mathbb{P}^1 & \mathbb{C}(z) \arrow[u, hook, "d"'] & \supseteq & \mathbb{C}[z] \arrow[u, hook, "?\ \text{might be bad!}"'] \end{tikzcd}$$

Can be fixed by taking the "integral closure of $\mathbb{C}[C]$ in $\mathbb{C}(C)$".

$$\begin{tikzcd} \mathbb{C}(C) & \supseteq & \mathcal{O} = \mathbb{C}[\tilde{C}] \\ \mathbb{C}(z) \arrow[u, hook] & \supseteq & \mathbb{C}[z] \arrow[u, hook, "\text{resolution of singularities.}"'] \end{tikzcd}$$

This picture came from number theory:

$$\begin{tikzcd} \mathbb{Q}(\sqrt{-7}) & \supseteq & \mathbb{Z}[\sqrt{-7}] \\ \mathbb{Q} \arrow[u, hook, "2"] & \supseteq & \mathbb{Z} \arrow[u, hook, "\text{might be bad!}"'] \end{tikzcd}$$

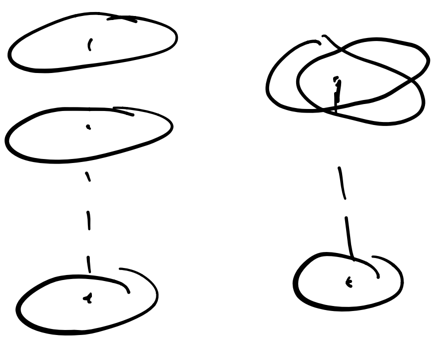
Dedekind tells us to take the integral closure of $\mathbb{Z}[\sqrt{-7}]$ in $\mathbb{Q}(\sqrt{-7})$:

$$\begin{array}{ccccl} \mathbb{Q}(\sqrt{-7}) & \supseteq & \mathcal{O} & = & \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]. \\ \cup\!\mid & & \cup\!\mid & & \\ \mathbb{Q} & \supseteq & \mathbb{Z} & & \end{array}$$

unique factorization of ideals.

Analogy:

points of C = prime ideals of $\mathbb{C}[C]$.



primes of $\mathbb{C}[C]$.

primes of $\mathbb{C}[z] = \mathbb{C}$

points of $\operatorname{Spec} \mathcal{O}_K$ = prime ideals P

$\downarrow$

points of $\operatorname{Spec} \mathbb{Z}$ = prime ideals p

We can also use the language

points = valuations.