

Concept of a Noetherian Ring: TFAE:

(1) Ideals of R are finitely generated.

$$I = a_1 R + a_2 R + \dots + a_n R.$$

(2) A.C.C.: Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ stabilizes:

$$\exists n \text{ such that } I_n = I_{n+1} = \dots.$$

(3) Every non-empty set of ideals contains a maximal element.

Proof: (1) \Rightarrow (2): Assume for contradiction

that $\exists I_1 \subsetneq I_2 \subsetneq \dots$.

Let $J = \bigcup_k I_k$, which is an ideal.

By hypothesis, $J = a_1 R + \dots + a_n R$.

Also: $\exists k$ such that $a_1, \dots, a_n \in I_k$.

$$\Rightarrow J = a_1 R + \dots + a_n R$$

$$\subseteq I_k \subsetneq I_{k+1} \subseteq J.$$

Contradiction. \parallel

(2) \Rightarrow (1): Assume for contradiction

that $I \in \mathcal{R}$ is not finitely generated.

Choose $a_1 \in I$. Since $I = a_1 R$,

choose $a_2 \in I \setminus a_1 R$

\vdots
 $a_k \in I \setminus (a_1 R + \dots + a_{k-1} R)$.

Thus we obtain an infinite ascending

chain: $a_1 R + \dots + a_{k-1} R \subsetneq a_1 R + \dots + a_k R$. \equiv

(2) \Rightarrow (3): Assume for contradiction,

\exists ^{nonempty} set of ideals S with no max.

element. Pick $I_1 \in S$. Since I_1 not maximal in S , pick $I_1 \subsetneq I_2 \subsetneq \dots$, to get infinite ascending chain.

(3) \Rightarrow (2): Consider any chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

Let $S = \{I_1, I_2, \dots\}$. By hypothesis,

S contains a max element, say $I_n \in S$.

Thus: $I_n = I_{n+1} = I_{n+2} = \dots$ \equiv

"Noetherian" = abstract substitute for well-ordering principle.



Hilbert's Basis Theorem:

R Noetherian $\implies R[x]$ Noetherian

Proof: Let R Noetherian & assume $R[x]$ is not Noetherian. Say,

$I \subseteq R[x]$ is not finitely generated.

Choose any $0 \neq f_1(x) \in I$ of minimal degree. Since $I \neq f_1 R[x]$, we can choose $f_2(x) \in I \setminus f_1 R[x]$ of minimal degree

$f_k(x) \in I \setminus (f_1 R[x] + \dots + f_{k-1} R[x])$

of minimal degree, so that

$\deg(f_1) \leq \deg(f_2) \leq \dots$

To obtain a contradiction, let $a_k \in R$ be leading coeff. of $f_k(x)$.

Then I claim:

$$a_1 R + \dots + a_{k-1} R \subsetneq a_1 R + \dots + a_k R$$

for all k , contradicting the fact that R is Noetherian.

So assume for contradiction that

$$a_k \in a_1 R + \dots + a_{k-1} R$$

$$a_k = a_1 b_1 + \dots + a_{k-1} b_{k-1} \quad (b_1, \dots, b_{k-1} \in R).$$

Define:

$$g_k(x) = f_k - b_1 x^{\deg f_k - \deg f_1} f_1 - b_2 x^{\deg f_k - \deg f_2} f_2 - \dots - b_{k-1} x^{\deg f_k - \deg f_{k-1}} f_{k-1} \in R[x]$$

Key Observations:

• $\deg(g_k) < \deg(f_k)$ because

leading coeff is

$$a_k - a_1 b_1 - a_2 b_2 - \dots - a_{k-1} b_{k-1} = 0.$$

- $g_k \in \mathbb{I} \setminus (f_1 R[x] + \dots + f_{k-1} R[x]),$

contradicting minimality of $\deg(f_k)$.

Indeed, if this is not the case, i.e., if $g_k \in f_1 R[x] + \dots + f_{k-1} R[x],$ then

$$f_k = g_k + b_1 \dots f_1 + \dots \text{stuff}$$

$$\in f_1 R[x] + \dots + f_{k-1} R[x],$$

contradicting the def. of f_k . \equiv

That was TRICKY!

Geometric Version: Any intersection of hypersurfaces can be expressed as an intersection of finitely many hypersurfaces.

Proof: for any ideal $\mathbb{I} \subseteq F[\vec{x}]$ we define the set of points

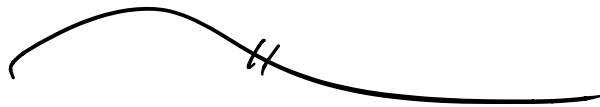
$$V(\mathbb{I}) = \{ \vec{p} : f(\vec{p}) = 0 \forall f \in \mathbb{I} \}$$

Thus $V(I) = \bigcap_{f \in I} V_f$ is an intersection of infinitely many hypersurfaces.

However, from H.B.T. we know that

$$I = f_1 \mathbb{F}[\vec{x}] + \dots + f_m \mathbb{F}[\vec{x}], \text{ hence}$$

$$V(I) = V_{f_1} \cap \dots \cap V_{f_m}.$$



We can restate the NSS in modern language:

(Weak): Let $I \subseteq \mathbb{F}[\vec{x}]$ be an ideal.

$$V(I) = \emptyset \Rightarrow I = \mathbb{F}[\vec{x}] \\ (\text{i.e., } 1 \in I)$$

$$[V_{f_1} \cap \dots \cap V_{f_m} \neq \emptyset \Rightarrow 1 = f_1 \hat{f}_1 + \dots + f_m \tilde{f}_m]$$

(Strong): If g vanishes on $V(I)$

then $g^r \in I$ for some $r \geq 0$.

More modern:

$$I(V(I)) = \sqrt{I}$$

I.O.U. the definitions of these



We've talked about points & hypersurfaces. What about shapes of intermediate dimension.

Definition: An affine (projective) variety is an intersection of (finitely many) affine (projective) hypersurfaces.

At least algebraically, these are not so hard to describe:

varieties in \mathbb{F}^n \longleftrightarrow radical ideals in $\mathbb{F}[x_1, \dots, x_n]$

irreducible varieties \longleftrightarrow prime ideals.

Two Extreme Cases:

① Minimal Prime Ideals = Irreducible Hypersurfaces

② Maximal (Prime) Ideals = points.

Proof: ① Is an abstract version of Study's Lemma. Let R be UFD.

Then I claim:

minimal primes = principal primes.

Let $0 \neq pR \subseteq R$ be principal prime.

Let $0 \neq Q \subseteq pR$ be prime, Pick

nonzero, nonunit $f \in Q$, Factor

into irreducibles:

$$f = p_1 p_2 \cdots p_k.$$

Since Q is prime, have $p_i \in Q$
for some i , hence $p_i R \subseteq Q \subseteq pR$.

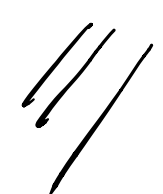
Since p_i irreducible, $p \mid p_i$,
 $p_i \not\sim 1$, have $p_i \sim p$, hence

$$p_i R = Q = pR. \quad \checkmark$$

Conclusion: pR is minimal prime.

Conversely, let $0 \neq P$ be minimal
prime. Choose nonzero, nonunit $f \in P$
and factor $f = p_1 p_2 \cdots p_k$. Since P
is prime, $p_i \in P$ for some i , hence

$0 \neq p_i R \subseteq P$. Since R UFD,
irred. p_i is prime hence $p_i R$ is
prime which implies $p_i R = P$ by
minimality of P .

Conclusion: P is principal. 

It turns out that (2) is actually equivalent to the Nullstellensatz!

We'll discuss this next time.