Concept of a Noetherian Ring: TFAE:

(1) Ideals of R are finitely generated.

$I = a_1 R + a_2 R + \cdots + a_n R$.

(2) A.C.C.: Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ stabilizes:

$\exists\, n$ such that $I_n = I_{n+1} = \cdots$.

(3) Every non-empty set of ideals contains a maximal element.

Proof: (1) $\Rightarrow$ (2): Assume for contradiction that $\exists\ I_1 \subsetneq I_2 \subsetneq \cdots$.

Let $J = \bigcup_k I_k$, which is an ideal.

By hypothesis, $J = a_1 R + \cdots + a_n R$.

Also: $\exists\ k$ such that $a_1, \ldots, a_n \in I_k$.

$$\Longrightarrow \quad J = a_1 R + \cdots + a_n R \subseteq I_k \subsetneq I_{k+1} \subseteq J.$$

Contradiction. ///

(2) $\Rightarrow$ (1): Assume for contradiction

that $I \subseteq R$ is not finitely generated.

Choose $a_1 \in I$. Since $I \neq a_1 R$,

choose $a_2 \in I \setminus a_1 R$

$\vdots$

$$a_k \in I \setminus (a_1 R + \cdots + a_{k-1} R).$$

Thus we obtain an infinite ascending chain: $a_1 R + \cdots + a_{k-1} R \subsetneq a_1 R + \cdots + a_k R$. ///

(2) $\Rightarrow$ (3): Assume for contradiction, $\exists$ nonempty set of ideals S with no max. element. Pick $I_1 \in S$. Since I_1 not maximal in S, pick $I_1 \subsetneq I_2 \subsetneq \cdots$, to get infinite ascending chain.

(3) $\Rightarrow$ (2): Consider any chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

Let $S = \{I_1, I_2, \ldots\}$. By hypothesis, S contains a max element, say $I_n \in S$.

Thus: $I_n = I_{n+1} = I_{n+2} = \cdots$ ///

"Noetherian" = abstract substitute for well-ordering principle.

Hilbert's Basis Theorem:

R Noetherian $\Longrightarrow$ $R[x]$ Noetherian

<u>Proof</u>: Let R Noetherian & assume $R[x]$ is not Noetherian. Say, $I \leq R[x]$ is not finitely generated.

Choose any $0 \neq f_1(x) \in I$ of minimal degree. Since $I \neq f_1 R[x]$, we can choose $f_2(x) \in I \setminus f_1 R[x]$ of minimal degree ....

$$f_k(x) \in I \setminus \left(f_1 R[x] + \cdots + f_{k-1} R[x]\right)$$

of minimal degree, so that

$$\deg(f_1) \leq \deg(f_2) \leq \cdots.$$

To obtain a contradiction, let $a_k \in R$ be leading coeff. of $f_k(x)$.

Then I claim:

$$a_1R + \cdots + a_{k-1}R \subsetneq a_1R + \cdots + a_kR$$

for all k, contradicting the fact that R is Noetherian.

So assume for contradiction that

$$a_k \in a_1R + \cdots + a_{k-1}R$$

$$a_k = a_1b_1 + \cdots + a_{k-1}b_{k-1} \quad (b_1, \ldots, b_{k-1} \in R).$$

Define:

$$\begin{aligned} g_k(x) = f_k &- b_1 x^{\deg f_k - \deg f_1} f_1 \\ &- b_2 x^{\deg f_k - \deg f_2} f_2 \\ &\vdots \\ &- b_{k-1} x^{\deg f_k - \deg f_{k-1}} f_{k-1} \in R[x] \end{aligned}$$

Key Observations:

- $\deg(g_k) < \deg(f_k)$ because leading coeff is

$$a_k - a_1b_1 - a_2b_2 - \cdots - a_{k-1}b_{k-1} = 0.$$

- $g_k \in I \setminus \left(f_1 R[x] + \cdots + f_{k-1} R[x]\right)$,

contradicting minimality of $\deg(f_k)$.

Indeed, if this is not the case, i.e., if $g_k \in f_1 R[x] + \cdots + f_{k-1} [x]$, then

$$f_k = g_k + b_1 \cdots f_1 \cdots \text{stuff}$$
$$\in f_1 R[x] + \cdots + f_{k-1} R[x],$$

contradicting the def. of f_k. ///

That was TRICKY!

Geometric Version: Any intersection of hypersurfaces can be expressed as an intersection of <u>finitely many</u> hypersurfaces.

<u>Proof</u>: For any ideal $I \subseteq F[\vec{x}]$ we define the set of points

$$V(I) := \{\vec{p} : f(\vec{p}) = 0 \ \forall f \in I\}$$

Thus $V(I) = \bigcap_{f \in I} V_f$ is an intersection of infinitely many hypersurfaces.

However, from H.B.T. we know that

$I = f_1 \mathbb{F}[\vec{x}] + \cdots + f_m \mathbb{F}[\vec{x}]$, hence

$$V(I) = V_{f_1} \cap \cdots \cap V_{f_m}.$$

We can restate the NSS in modern language:

(Weak): Let $I \leq \mathbb{F}[\vec{x}]$ be an ideal.

$$V(I) = \emptyset \implies I = \mathbb{F}[\vec{x}]$$

(i.e., $1 \in \mathbb{F}[\vec{x}]$)

$$\left[V_{f_1} \cap \cdots \cap V_{f_m} \neq \emptyset \implies 1 = f_1 \tilde{f}_1 + \cdots + f_m \tilde{f}_m \right]$$

(Strong): If g vanishes on $V(I)$ then $g^r \in I$ for some $r \geq 0$.

More modern:

$$I(V(I)) = \sqrt{I}$$

I.O.U. the definitions of these

We've talked about points & hypersurfaces. What about shapes of intermediate dimension.

Definition: An affine (projective) variety is an intersection of (finitely many) affine (projective) hypersurfaces.

At least algebraically, these are not so hard to describe:

$$\begin{array}{c} \text{varieties} \\ \text{in } \mathbb{F}^n \end{array} \longleftrightarrow \begin{array}{c} \text{radical ideals} \\ \text{in } \mathbb{F}[x_1, \dots, x_n] \end{array}$$

irreducible varieties $\longleftrightarrow$ prime ideals.

Two Extreme Cases:

(1) Minimal Prime Ideals = Irreducible Hypersurfaces

(2) Maximal (Prime) Ideals = points.

<u>Proof</u>: (1) Is an abstract version of Study's Lemma. Let R be UFD. Then I claim:

minimal primes = principal primes.

Let $0 \subsetneq pR \subsetneq R$ be principal prime. Let $0 \subsetneq Q \subseteq pR$ be prime, Pick nonzero, nonunit $f \in Q$, Factor into irreducibles:

$$f = p_1 p_2 \cdots p_k.$$

Since Q is prime, have $p_i \in Q$ for some i, hence $p_i R \subseteq Q \subseteq pR$. Since p_i irreducible, $p \mid p_i$, $p_i \not\sim 1$, have $p_i \sim p$, hence

$$p_i R = Q = pR. \quad \checkmark$$

Conclusion: pR is minimal prime.

Conversely, let $0 \subsetneq P$ be minimal prime. Choose nonzero, nonunit $f \in P$ and factor $f = p_1 p_2 \cdots p_k$. Since P is prime, $p_i \in P$ for some i, hence

$0 \subsetneq p_i R \subseteq P$. Since R UFD, irred. p_i is prime hence $p_i R$ is prime which implies $p_i R = P$ by minimality of P.

Conclusion: P is principal. ///

It turns out that 2 is actually equivalent to the Nullstellensatz!

We'll discuss this next time.