Relation between affine & projective varieties. Let $\mathbb{F}$ be algebraically closed.

On the geometric side: We'll identify $\mathbb{F}^n \subseteq \mathbb{FP}^n$ with the affine chart $U_{n+1} = \mathbb{FP}^n \setminus H_{n+1}$.

Let $V \subseteq \mathbb{F}^n$ be an affine variety. In general $V \subseteq \mathbb{FP}^n$ is <u>not</u> a projective variety, so we define

$$\overline{V} = \text{projective Zariski closure of set } V \text{ in } \mathbb{FP}^n,$$

and call this the projective closure of V. Conversely, given projective variety $V \subseteq \mathbb{FP}^n$, I claim that $V \cap \mathbb{F}^n \subseteq \mathbb{F}^n$ is an affine variety. Indeed, we already know from Study's Lemma that this holds for hypersurfaces. [Dehomogenize the polynomial.]

Hence the same holds for all intersections of hypersurfaces, i.e., for all varieties.

How do these operations translate into the language of ideals?

Cone over an affine variety.

Given set $S \subseteq \mathbb{F}^{n+1}$ we define the cone

$$\text{Cone}(S) = \{\lambda\vec{p} : \vec{p} \in S, \lambda \in \mathbb{F}\}.$$

This is the smallest conical set containing S. If $V = V(\underbrace{I}_{\text{radical}})$ is a variety then I claim that

$$I' := I(\text{Cone}(V)) = \text{sub-ideal of } I \text{ generated by its homogeneous elements.}$$

Furthermore, $I' \subseteq I$ is the largest homogeneous ideal contained in I.

Since the rad. of hom. ideal is hom.

This implies that I' is radical.

And it follows that the Zariski closure

$VI(\mathrm{Cone}(V)) = V(I')$ is the smallest

conical ideal containing V.

[Remark: If V is a variety, then

$\mathrm{Cone}(V)$ is not necessarily a variety.

For example, let $V = \{(x,y) : y = 1\} \subseteq \mathbb{F}^2$.

Then $\mathrm{Cone}(V) = \{(x,y) : y \neq 0\} \cup \{(0,0)\}$.

Exercise: This is not a variety.]

Proof: First we show

$$I(\mathrm{Cone}(V)) = I' := \text{gen by hom. elements.}$$

Indeed, if $f \in I(\mathrm{Cone}(V))$ then since

$\mathrm{Cone}(V)$ is conical, $I(\mathrm{Cone}(V))$ is homo.

hence $f^{(k)} \in I(\mathrm{Cone}(V))$ for all k.

We also have $I(\mathrm{Cone}(V)) \subseteq I = I(V)$.

so that $f^{(k)} \in I \;\; \forall k$.

It follows that $f^{(k)} \in I'$

$\Longrightarrow f = \sum F^{(k)} \in I'$.

We have shown that $I(\mathrm{Cone}(V)) \subseteq I'$

Conversely, let $f \in I'$, so $f = \sum F_i g_i$

where $F_i \in I$ are homogeneous.

If $\vec{p} \in V$ we want to show that

$f(\lambda \vec{p}) = 0 \quad \forall \lambda$ so f vanishes on

the cone (V), hence $f \in I(\mathrm{Cone}(V))$.

well, since $F_i = I = I(V)$ we know that

$F_i(\vec{p}) = 0$. Since F_i is homogeneous,

$F_i(\lambda \vec{p}) = 0$, and finally

$$f(\lambda \vec{p}) = \sum F_i(\lambda \vec{p})\, g_i(\lambda \vec{p}) = 0. \quad \checkmark$$

Next: Show $I' \subseteq I$ is the largest homogeneous sub-ideal.

Suppose $J \subseteq I$ is homogeneous, so

$J = F_1 \mathbb{F}[\vec{x}] + \cdots + F_m \mathbb{F}[\vec{x}]$

for homogeneous $F_1, \ldots, F_m \in J \subseteq I$.

But then $F_i \in I'$, so that $J \subseteq I'$.

[Corollary: If I is radical then we see that I' is radical. Proof: If $g^r \in I'$ then $g^r \in I \Rightarrow g \in I$, so $\sqrt{I'} \subseteq I$. But I' hom. $\Rightarrow \sqrt{I'}$ hom. $\Rightarrow I' = \sqrt{I'}$ by maximality of I'.]

To finish the proof, let $W \supseteq V = V(I)$ be any conical variety containing V. We want to show $W \supseteq VI(\mathrm{Cone}(V))$

[Note: $VI(\mathrm{Cone}(V)) = V(I')$.]

We want to show $W \supseteq V(I')$.

To show this apply I to $W \supseteq V(I)$ to get $I(W) \subseteq IV(I) = I$.

Since $I(W)$ is homogeneous [W conical] this implies $I(W) \subseteq I'$ by the previous step. Finally, apply V to get

$$V(I') \subseteq V(I(W)) = W.$$

Special Case: Projective closure of an affine variety. Let $\vec{x} = (x_1, \dots, x_{n+1})$ & $\vec{x}' = (x_1, \dots, x_n)$. Let $V \subseteq \mathbb{F}^n$ be affine variety with $V = V(I)$ for some ideal $I \leq \mathbb{F}[\vec{x}']$, say

$$I = f_1 \mathbb{F}[\vec{x}'] + \dots + f_m \mathbb{F}[\vec{x}'] \leq \mathbb{F}[\vec{x}']$$

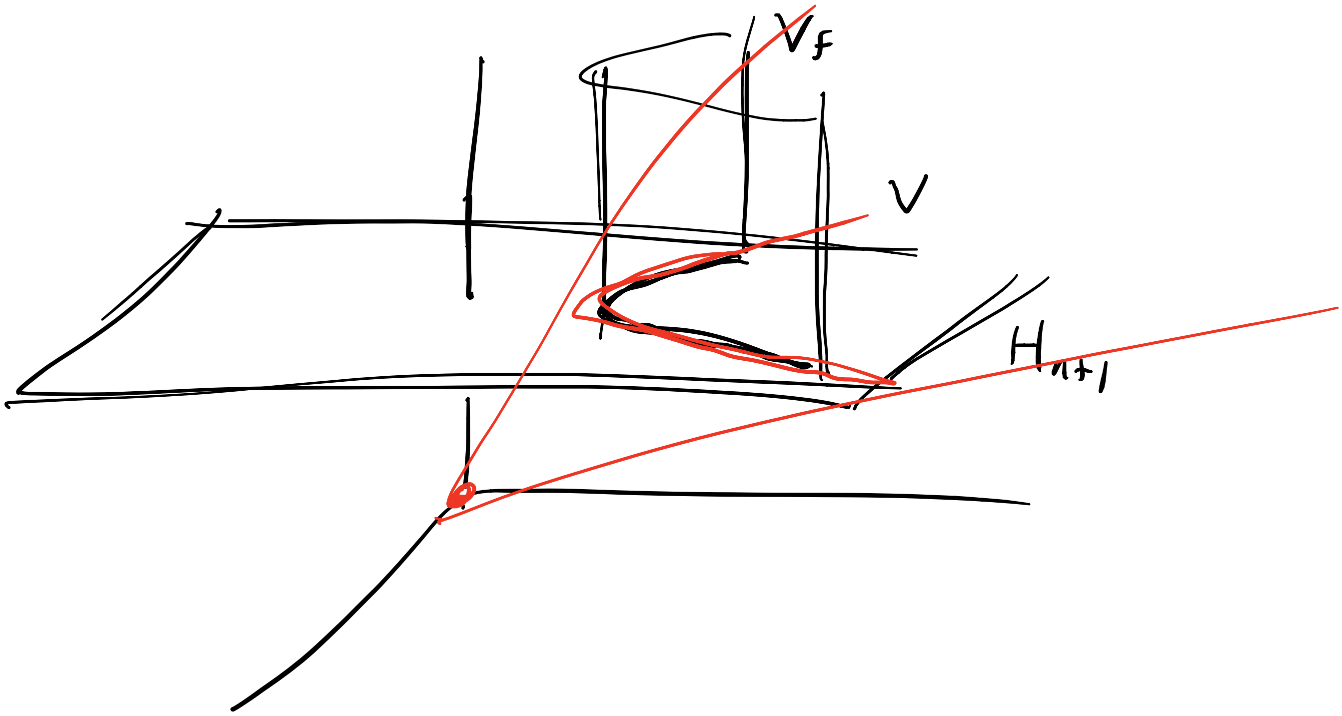
Define $J = f_1 \mathbb{F}[\vec{x}] + \dots + f_m \mathbb{F}[\vec{x}] \leq \mathbb{F}[\vec{x}]$.

So the ideal of $V \subseteq \mathbb{F}^{n+1}$ is

$$J + (x_{n+1} - 1)\mathbb{F}[\vec{x}].$$

Indeed, we can view V as the intersection of hypersurfaces V_{f_i} & hyperplane H_{n+1}.

Picture:



Therefore the projective closure

$$\overline{V} = V I(\operatorname{Cone}(V))$$

is defined by the ideal

$$I^* := \left(J + (x_{n+1} - 1)\mathbb{F}[\vec{x}] \right)'$$

I claim there is an easier way to describe this:

I^* = ideal generated by homogenizations of elements of I.

$$= \left\langle \{ f^* : f \in I \} \right\rangle \subseteq \mathbb{F}[\vec{x}].$$

We call this the "homogenization of I".

$$\begin{array}{ccc} \mathbb{F}[\vec{x}'] & \le & \mathbb{F}[\vec{x}] \\ \cup\!| & & \cup\!| \\ I & \rightsquigarrow & I^* \end{array}$$

Proof: We need to show that

$$I^* = I(\mathrm{Cone}(V)).$$

One direction: If $f \in I^*$ then

$f = \sum (f_i)^* g_i$ for some $f_i \in I$.

Since f_i vanishes on $V = V(I)$, then

$(f_i)^*$ vanishes on $\mathrm{Cone}(V)$.

$\Rightarrow$ f vanishes on $\mathrm{Cone}(V)$

$\Rightarrow$ $f \in I(\mathrm{Cone}(V))$.

Conversely, to show $I(\mathrm{Cone}(V)) \subseteq I^*$.

$I(\mathrm{Cone}(V))$ is homogeneous.

$\Rightarrow$ generated by finitely many homogeneous polynomials $F_1, --, F_m$.

F_i vanishes on $\mathrm{Cone}(V)$

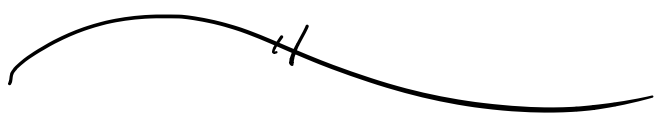
$(F_i)_*$ vanishes on V.

$\Longrightarrow (F_i)_* \in I(V)$.

Observe $x_{n+1}^e F_i = ((F_i)_*)^*$

We want $F_i \in I^*$.

[I guess since $\mathrm{Cone}(V) \not\subseteq H_{n+1}$ we can choose F_i so $x_{n+1} \nmid F_i$.] .



Too much theory.

Example: Twisted Cubic.