Algebra Qualifying Exam **Spring 2016**
List of Topics **Drew Armstrong**

Posets and Lattices.

- Limits: $\wedge$, 1
- Colimits: $\vee$, 0
- Examples: Divisors, Subsets, Subgroups, Submodules
- Modular Elements vs. Normal Subgroups
- Galois Connections: $p \leq q^* \Longleftrightarrow q \leq p^*$
- Galois Connection $2^S \rightleftarrows 2^T$ from a Relation $R \subseteq S \times T$.
- Example: Image and Preimage of a Homomorphism

$$\mathscr{L}(G, \ker \varphi) \cong \mathscr{L}(\operatorname{im} \varphi)$$

Equivalence Relations and Quotients.

- Definition of Equivalence $\sim \subseteq S \times S$
- Universal Property of the Quotient $S/\sim$
- Equivalence from a Function $f : S \to T$

$$s_1 \sim s_2 \Longleftrightarrow f(s_1) = f(s_2)$$

- Canonical Factorization

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[rrr, bend left, "f"] & (S/\sim) \arrow[r, leftrightarrow] & \operatorname{im} f \arrow[r, hook] & T \end{tikzcd}$$

- G-Invariant Relations vs. Normal Subgroups
- 1st, 2nd, 3rd Isomorphism Theorems for Groups and Modules
- Jordan–Hölder Theorem
- Examples: Cyclic Groups, Vector Spaces

Products of Groups.

- Multiplication Function $\mu : H \times K \to G$ for Subgroups $H, K \subseteq G$
- μ Injective $\Longleftrightarrow H \cap K = 1$
- $\operatorname{im} \mu$ is Subgroup $\Longleftrightarrow HK = KH \Longleftarrow H \subseteq N_G(K)$ OR $K \subseteq N_G(H)$
- Semidirect Product: $H \subseteq N_G(K)$ XOR $K \subseteq N_G(H)$
- Direct Product: $H \subseteq N_G(K)$ AND $K \subseteq N_G(H)$
- External vs. Internal Direct and Semidirect Products
- Semidirect: Right-Split Short Exact Sequence
- Direct: Left-Split Short Exact Sequence
- Examples: Dihedral Groups, General Affine Groups

Automorphism Groups.

- Symmetric Group: $S_n = \operatorname{Aut}_{\mathsf{Set}}(\{1, 2, \ldots, n\})$
- General Linear Group: $\mathrm{GL}_n(K) = \operatorname{Aut}_{K\text{-}\mathsf{Mod}}(K^n)$
- Alternating and Special Subgroups
- Abel–Galois–Ruffini: S_5 is not Solvable
- A_n and $\mathrm{PSL}_n(K)$ are (usually) Simple

$$|\mathrm{PSL}_n(\mathbb{F}_q)| = \frac{q^{\binom{n}{2}}(q^2 - 1)(q^3 - 1) \cdots (q^n - 1)}{\gcd(n, q - 1)}$$

Category of G-Sets.

- Definitions: Category, Functor, Natural Transformation
- Definition of "Group" Models $\mathrm{Aut}_{\mathcal{C}}(-)$
- G-Set is $G \to \mathrm{Aut}_{\mathsf{Set}}(X)$, so G-$\mathsf{Set} = \mathsf{Set}^G$
- Orbits and Stabilizers
- Fundamental Theorem of G-Sets
- Examples: Cosets, Double Cosets, Grassmannians
- Conjugation and the Class Equation

Sylow Theory.

- Lagrange's Theorem
- Converse to Lagrange is Not True
- Cauchy's Theorem: $p \mid |G| \implies G$ has an element of order p
- Sylow Theorem Parts 1, 2, 3
- Application: Groups of Size $p^\alpha q$ and pqr, for Prime p, q, r
- Application: Primary Decomposition of Finite Abelian Groups

Differences Between Grp and Ab.

- Definitions: Zero Object, Kernel, Cokernel, Monomorphism, Epimorphism
- Ab Has a Biproduct: $\oplus$
- Ab is Enriched Over Ab
- $\mathrm{End}_{\mathsf{Ab}}(-) : \mathsf{Ab} \to \mathsf{Rng}$
- Multiplication is Repeated Addition: $\mathbb{Z} = \mathrm{End}_{\mathsf{Ab}}(\mathbb{Z})$

Rings and R-Modules.

- Definition of "Ring" Models $\mathrm{End}_{\mathsf{Ab}}(-)$
- R-Module is $R \to \mathrm{End}_{\mathsf{Ab}}(M)$, so R-$\mathsf{Mod} = \mathsf{Ab}^R$
- $\mathbb{Z}$-$\mathsf{Mod} = \mathsf{Ab}$
- K-$\mathsf{Mod} = K$-Vec
- Definitions: Free Modules, Adjoint Functors, Meta-Theorem (RAPL)
- $\mathrm{Forget} : R$-$\mathsf{Mod} \to \mathsf{Ab}$ has Left Adjoint $\mathrm{Free}_R(A) = R^{\oplus A}$
- Application: $R^{\oplus(A \sqcup B)} = R^{\oplus A} \oplus R^{\oplus B}$

Algebras Over a Commutative Ring.

- Definitions: R-Alg and R-CAlg
- Polynomials $R[X]$ = Free Commutative Algebra
- Evaluation Homomorphism $\varphi_a : R[x] \to S$
- Algebraic vs. Transcendental

Modules Over a PID, Part I.

- Field $\Rightarrow$ Euclidean $\Rightarrow$ PID $\Rightarrow$ UFD
- $\mathbb{Z}$ and $K[x]$ are Euclidean
- Dimension Exists in K-Mod (Steinitz Exchange)
- Localization of a Module: $M \to A^{-1}M = M \otimes_R A^{-1}R$
- Rank Exists in R-Mod When R is an Integral Domain
- Submodule of $R^{\oplus A}$ is Free of Rank $\leq |A|$ when R is PID
- FTFGMPID, Part I:
$$M \cong R^{\oplus \mathrm{rank}(M)} \oplus \mathrm{Tor}_R(M)$$

Matrix Notation (Assume R is Commutative).

- $\varphi : \bigoplus M_j \to \bigoplus N_i$ is Determined by Components $\varphi_{ij} : M_j \to N_i$
- $\varphi : R^{\oplus m} \to R^{\oplus n}$ is Determined by Matrix $[\varphi] \in \mathrm{Mat}_{n\times m}(R)$
- R-Mod is Enriched Over R-Mod
- Choosing a Basis is Isomorphism of R-Modules: $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^{\oplus m}, R^{\oplus n}) \xrightarrow{\sim} \mathrm{Mat}_{n\times m}(R)$
- Special Case: Isomorphism of R-Algebras: $\mathrm{End}_{R\text{-}\mathsf{Mod}}(R^{\oplus n}) \xrightarrow{\sim} \mathrm{Mat}_n(R)$
- Change of Basis: $\mathrm{GL}_n(R) \times \mathrm{GL}_m(R)$ acts on $\mathrm{Mat}_{n\times m}(R)$ by $(A, B) \cdot C = ACB^{-1}$
- Special Case: $\mathrm{GL}_n(R)$ acts on $\mathrm{Mat}_n(R)$ by Conjugation

Modules Over a PID, Part II.

- Elementary Matrices $E_{ij}(r)$, $E_{ii}(r)$, P_{ij}
- RREF Exists Over a Field
- Elementary Matrices Generate $\mathrm{GL}_n(R)$ when R is Euclidean
- Pseudo-Elementary Matrices Generate $\mathrm{GL}_n(R)$ when R is PID
- Smith Normal Form Over a PID
- Chinese Remainder Theorem Over a PID
- FTFGMPID, Part II:

$$\mathrm{Tor}_R(M) \cong \bigoplus \frac{R}{(f_i)} \cong \bigoplus \frac{R}{(p_i^{\alpha_{ij}})}$$

Applications of FTFGMPID.

- Fundamental Theorem of Finitely Generated Abelian Groups
- Primitive Root Theorem
- $K[x]$-Modules = Pairs (V, φ) with $V \in K$-Vec and $\varphi \in \mathrm{End}_{K\text{-}\mathsf{Vec}}(V)$
- Rational Canonical Form
- Minimal vs. Characteristic Polynomial
- Jordan Canonical Form
- Jordan-Chevalley Decomposition