Math 762 **Spring 2016**
Homework 1 **Drew Armstrong**

Problem 1. Infinite Products and Coproducts in Ab. We have seen that finite products and coproducts agree in Ab. However, the same is not true for **infinite** products and coproducts. Let I be a set and let $\{A_i\}_{i\in I}$ be a family of abelian groups, each equal to some fixed group A.

(a) Show that the set $A^I := \mathrm{Hom}_{\mathsf{Set}}(I, A)$ is an abelian group. Furthermore, show that we can think of this group as the infinite product $\Pi_{i\in I}A_i$ in the category Ab.
(b) Let $A^{\oplus I}$ denote the subgroup of A^I in which **all but finitely many** elements of I are sent to the identity element $0 \in A$. Show that we can think of $A^{\oplus I}$ as the infinite coproduct $\bigoplus_{i\in I} A_i$ in the category Ab.
(c) Show that the inclusion $A^{\oplus I} \subseteq A^I$ can be strict. [Hint: Let $A = \mathbb{Z}/10\mathbb{Z}$ and $I = \mathbb{Z}$.]

Problem 2. What is a polynomial? Let $(M, \cdot, 1_M)$ be a monoid and let $(R, +, \circ, 0_R, 1_R)$ be a ring. The monoid ring $R[M]$ is the abelian group $R^{\oplus M}$ together with the following operation: for all $a, b \in R[M]$ and $m \in M$ we define $a * b \in R[M]$ by the formula

$$(a * b)_m := \sum_{m_1\cdot m_2=m} a_{m_1} \circ b_{m_2}.$$

Note that the sum on the right exists because $a_{m_1} \circ b_{m_2} = 0_R$ for all but finitely many pairs $(m_1, m_2) \in M^2$. One can check (you don't need to) that this defines a ring structure on $R[M]$.

(a) Show that there is an obvious injective ring homomorphism $R \hookrightarrow R[M]$.
(b) Thinking of $(\mathbb{N}, +, 0)$ as a monoid, prove that the monoid ring $R[\mathbb{N}]$ is isomorphic to the polynomial ring in one variable $R[x]$. [Remark: In fact, we could think of $R[\mathbb{N}]$ as the **definition** of the polynomial ring. I mean, what *is* x anyway?]

Problem 3. Evaluation of Polynomials. Let $\varphi : R \to S$ be a ring homomorphism and assume that the image of φ is in the center of S:

$$\mathrm{im}\,\varphi \subseteq Z(S) := \{t \in S : st = ts \text{ for all } s \in S\}.$$

(a) For all $s \in S$ prove that **there exists a unique ring homomorphism** $\varphi_s : R[x] \to S$ satisfying $\varphi_s(x) = s$ and $\varphi_s(r) = \varphi(r)$ for all $r \in R$ (thought of as a subring of $R[x]$ via Problem 2(a)). [Remark: When $R \subseteq S$ is a subring with inclusion homomorphism $i : R \hookrightarrow S$ we refer to the map $i_s : R[x] \to S$ as evaluation at s.]
(b) Show that the result of part (a) can fail when the image of φ is **not** in the center of S. [Remark: This is the place where the theories of commutative and noncommutative rings begin to diverge.]

The next two problems illustrate an important difference between commutative and noncommutative rings.

Problem 4. Descartes' Theorem. Let R be a **commutative ring** and for all $\alpha \in R$ consider the evaluation morphism $i_\alpha : R[x] \to R$ from Problem 3. For simplicity we will use the notation "$f(\alpha)$" $:= i_\alpha(f(x))$.

(a) Given $f(x) \in R[x]$ and $\alpha \in R$, prove that we have $f(\alpha) = 0$ if and only if $f(x) = (x - \alpha)g(x)$ for some $g(x) \in R[x]$. [Hint: Use division with remainder.]

(b) If R is, furthermore, an integral domain (i.e., if $ab = 0$ implies $a = 0$ or $b = 0$) then the degree function $\deg : R[x] \setminus \{0\} \to \mathbb{N}$ satisfies $\deg(fg) = \deg(f) + \deg(g)$. Use this fact to prove that a polynomial of degree n over an integral domain has at most n distinct roots. [Hint: Use part (a) and induction.]

Problem 5. The Original Noncommutative Ring. The ring (actually an $\mathbb{R}$-algebra) of quaternions was defined by William Rowan Hamilton on the 16th of October, 1843. He defined it as the 4-dimensional $\mathbb{R}$-vector space

$$\mathbb{H} := \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\},$$

where the abstract basis elements $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}.$$

(a) Prove that $\mathbb{H}$ can be realized as a subring (actually an $\mathbb{R}$-subalgebra) of the ring of 2×2 matrices over $\mathbb{C}$. [Hint: Let $i \in \mathbb{C}$ be the imaginary unit. Show that the $\mathbb{R}$-linear map defined on the basis by

$$\mathbf{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is injective. Then show that the relations are satisfied.]

(b) Use part (a) to compute the center $Z(\mathbb{H})$.

(c) It seems that the polynomial $x^2 + \mathbf{1} \in \mathbb{H}[x]$ of degree 2 has at least **three** distinct roots: $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$. What's the problem?

Problem 6. Monomorphisms and Epimorphisms. The notions of injective and surjective functions are not categorically well-behaved. In a general category they should be replaced with the notions of "monomorphism" and "epimorphism".

Let $\alpha : X \to Y$ be a morphism in a category $\mathscr{C}$. We say that α is a monomorphism if for all objects $Z \in \mathscr{C}$ and all morphisms $\beta_1, \beta_2 : Z \to X$ we have

$$\alpha \circ \beta_1 = \alpha \circ \beta_2 \quad \Longrightarrow \quad \beta_1 = \beta_2.$$

We say α is an epimorphism if for all $Z \in \mathscr{C}$ and $\beta_1, \beta_2 : Y \to Z$ we have

$$\beta_1 \circ \alpha = \beta_2 \circ \alpha \quad \Longrightarrow \quad \beta_1 = \beta_2.$$

(a) In the category Set, prove that monomorphisms are the same as injective functions and epimorphisms are the same as surjective functions.

(b) In the category Rng, prove that an epimorphism may fail to be surjective.