Math 762 **Spring 2016**
Final Exam **Drew Armstrong**

Problem 1. ***R*-Algebra Generalities.** Let R be a commutative ring.

(a) State the definition of an R-algebra.

An R-algebra is a pair (S, φ) where
- S is a ring,
- $\varphi : R \to S$ is a ring homomorphism satisfying

$$\operatorname{im}\varphi \subseteq Z(S) = \{s \in S : \forall\, t \in S, st = ts\}.$$

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(b) State the definition of a commutative R-algebra.

An R-algebra (S, φ) is called commutative when S is a commutative ring. In this case the condition $\operatorname{im}\varphi \subseteq Z(S)$ is vacuous, so a commutative R-algebra is the same as a homomorphism of commutative rings $\varphi : R \to S$. ///

(c) Let $R[A]$ denote the free commutative R-algebra generated by the set A. State its definition. (Such a thing exists, but please don't prove this.)

The free commutative R-algebra generated by the set A consists of a commutative R-algebra $R \to R[A]$ and a set function $A \to R[A]$ satisfying the following univeral property:

For all commutative R-algebras $R \to S$ and all set functions $A \to S$ there exists a unique ring homomorphism $\varphi : R \to S$ such that

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
R[A] \arrow[rr, dashed, "\exists!\varphi"] & & S \\
& R \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

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(d) Let R-CAlg be the category of commutative R-algebras and let R-Mod be the category of R-modules. State the definition of the "forgetful functor" $U : R\text{-}\mathsf{CAlg} \to R\text{-}\mathsf{Mod}$.

Given a commutative R-algebra $\varphi : R \to S$, we let $U(S)$ denote the R-module consisting of the pair $(|S|, \lambda)$, where $|S|$ is the underlying abelian group of S and λ is the ring homomorphism

$$\lambda : R \to \operatorname{End}_{\mathsf{Ab}}(|S|)$$

defined by $\lambda_r(s) := \varphi(r)s = s\varphi(r)$ for all $s \in |S|$. ///

(e) State what it means for the functor $F : R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{CAlg}$ to be left adjoint to U. (Such a functor exists, but don't prove this.)

We say that $F : R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{CAlg}$ is left adjoint to $U : R\text{-}\mathsf{CAlg} \to R\text{-}\mathsf{Mod}$ if we have a family of bijections

$$\tau_{M,S} : \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, U(S)) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}\mathsf{CAlg}}(F(M), S)$$

that is "natural" in the arguments $M \in R\text{-}\mathsf{Mod}$ and $S \in R\text{-}\mathsf{CAlg}$. ///

(f) Assume without proof that that $R[A] = F(R^{\oplus A})$ (which is true) and assume that "$\otimes_R$" is the name of the **coproduct** in the category $R\text{-}\mathsf{CAlg}$ (which is also true). In this case explain why we have an isomorphism of R-algebras:

$$R[A \sqcup B] \cong R[A] \otimes_R R[B].$$

The key fact is that left adjoint functors commute with colimits. Since coproducts are examples of colimits, and since the coproducts in $\mathsf{Set}, R\text{-}\mathsf{Mod}, R\text{-}\mathsf{CAlg}$ are $\sqcup, \oplus, \otimes_R$, respectively, we have the following chain of R-algebra isomorphisms:

$$\begin{aligned} R[A \sqcup B] &\cong F(R^{\oplus A \sqcup B}) \\ &\cong F(R^{\oplus A} \oplus R^{\oplus B}) \\ &\cong F(R^{\oplus A}) \otimes_R F(R^{\oplus B}) \\ &\cong R[A] \otimes_R R[B]. \end{aligned}$$

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Problem 2. Evaluation of Polynomials. Let R be a commutative ring and define $R[X] = R[x_1, \ldots, x_n]$ where $X = (x_1, \ldots, x_n)$ is an n-tuple of variables. For each $A \in R^n$ we will write $\varphi_A : R[X] \to R$ for the canonical evaluation map. Now for each "formal polynomial" $f(X) \in R[X]$ we can define a "polynomial function" $\varphi_f : R^n \to R$ by $\varphi_f(A) := \varphi_A(f(X)) = f(A)$. In summary, we have a function

$$\varphi : R[X] \to \mathrm{Hom}_{\mathsf{Set}}(R^n, R).$$

(a) Prove that φ is a ring homomorphism. [Hint: Don't do much.]

If we define a commutative ring structure on $\mathrm{Hom}_{\mathsf{Set}}(R^n, R)$ by "pointwise" addition and multiplication, then for all $f(X), g(X) \in K[X]$ and $A \in R^n$ we have

$$\varphi_{f+g}(A) = (f+g)(A) = f(A) + g(A) = \varphi_f(A) + \varphi_g(A) =: (\varphi_f + \varphi_g)(A)$$

and

$$\varphi_{fg}(A) = (fg)(A) = f(A)g(A) = \varphi_f(A)\varphi_g(A) =: (\varphi_f \cdot \varphi_g)(A),$$

hence it follows that $\varphi_{f+g} = \varphi_f + \varphi_g$ and $\varphi_{fg} = \varphi_f \cdot \varphi_g$. Then note that for all $A \in R^n$ we have $\varphi_1(A) = 1 = 1(A)$, so that $\varphi_1 = 1$. ///

[Remark: Maybe I did too much?]

(b) If R is an **infinite integral domain** and if $n = 1$, prove that φ is injective. [Hint: By part (a) you only need to show that $\ker \varphi = 0$. Use the fact (proved on HW1) that a polynomial $f(x) \in R[x]$ of degree m has at most m roots in R.]

Proof. Suppose that $f(x) \in \ker \varphi \subseteq R[x]$. This means that for all $a \in R$ we have $\varphi_f(a) = f(a) = 0$. Since R is infinite we have found infinitely many distinct roots of the polynomial $f(x)$. Since R is an integral domain, this implies that $f(x) = 0$. $\square$

(c) If R is an **infinite integral domain**, prove that φ is injective for any n. [Hint: Induction on part (b). Use the fact that $R[x_1, \ldots, x_{n-1}]$ is an infinite integral domain.]

Proof. Assume for induction that the map $\varphi : R[x_1, \ldots, x_{n-1}] \to \mathrm{Hom}_{\mathsf{Set}}(R^{n-1}, R)$ is injective. Now consider $f(X) = \sum_i g_i(x_1, \ldots, x_{n-1}) x_n^i \in R[X]$ and assume that $f(X) \in \ker \varphi \subseteq R[X]$, i.e., that $\varphi_f(A) = f(A) = 0$ for all $A = (a_1, \ldots, a_n) \in R^n$.

If we fix $(a_1, \ldots, a_{n-1})$, then as a_n ranges over R we see that $\sum_i g_i(a_1, \ldots, a_{n-1}) x_n^i \in R[x_n]$ has infinitely many roots in the integral domain R. By part (b) this implies that $g_i(a_1, \ldots, a_{n-1}) = 0$ for all i. Then as $(a_1, \ldots, a_{n-1})$ ranges over R^{n-1} we find that each $g_i(x_1, \ldots, x_{n-1})$ determines the zero function $R^{n-1} \to R$. By induction this implies that each $g_i(x_1, \ldots, x_{n-1})$ is the zero polynomial, hence $f(X) = 0$. $\square$

Problem 3. Persistence of Identities. Let R be a commutative ring and consider two matrices $A, B \in \mathrm{Mat}_n(R)$. When R is a field we know that $\det(AB) = \det(A)\det(B)$; in this problem you will prove that the same result holds without any hypothesis on R.

(a) Explain why the category $\mathbb{Z}$-CAlg of commutative $\mathbb{Z}$-algebras is just the category of commutative rings.

A commutative $\mathbb{Z}$-algebra is a pair (S, φ) where S is a commutative ring and $\varphi : \mathbb{Z} \to S$ is a ring homomorphism. But since $\mathbb{Z}$ is the initial object in the category of rings, the homomorphism φ is redundant. A morphism of $\mathbb{Z}$-algebras $(S_1, \varphi_1) \to (S_2, \varphi_2)$ is a ring homomorphism $\Phi : S_1 \to S_2$ such that $\Phi \circ \varphi_1 = \varphi_2$. But since φ_1 and φ_2 are redundant, Φ is just a ring homomorphism. ///

(b) Consider two n^2-tuples of variables $X = (x_{ij})$ and $Y = (y_{k\ell})$ and the commutative polynomial ring $\mathbb{Z}[X, Y]$ in $2n^2$ variables. For any two matrices $A = (a_{ij}), B = (b_{k\ell}) \in \mathrm{Mat}_n(R)$ explain why there exists a **unique ring homomorphism**

$$\varphi_{A,B} : \mathbb{Z}[X, Y] \to R$$

such that $\varphi_{A,B}(x_{ij}) = a_{ij}$ and $\varphi_{A,B}(y_{k\ell}) = b_{k\ell}$ for all $i, j, k, \ell \in \{1, \ldots, n\}$. [Hint: (a).]

Thinking of R as a $\mathbb{Z}$-algebra by part (a) and thinking of $\mathbb{Z}[X, Y]$ as the free $\mathbb{Z}$-algebra from Problem 1(c) gives us a unique "evaluation" $\mathbb{Z}$-algebra homomorphism. But, by part (a), $\mathbb{Z}$-algebra homomorphisms are just ring homomorphisms. ///

(c) Consider the formal polynomial $f(X, Y) := \det(XY) - \det(X)\det(Y) \in \mathbb{Z}[X, Y]$. Prove that for all matrices $A, B \in \mathrm{Mat}_n(R)$ we have $f(A, B) := \varphi_{A,B}(f(X, Y)) = 0$. [Hint: You can assume that this is true when R is a field. Use part (b) and Problem 2(c) to show that $f(X, Y)$ is actually the zero element of $\mathbb{Z}[X, Y]$.]

Proof. Let K be any infinite field. Since K is a field we have $f(A, B) = 0$ for all matrices $A, B \in \mathrm{Mat}_n(K)$. Then since K is an infinite domain, Problem 2(c) implies that $f(X, Y)$ is the zero element of $\mathbb{Z}[X, Y]$. Finally, for any commutative ring R and any matrices $A, B \in \mathrm{Mat}_n(R)$ we have

$$f(A, B) = \varphi_{A,B}(f(X, Y)) = \varphi_{A,B}(0) = 0.$$

$\square$

Problem 4. Modules over a PID. Let R be a PID and let $T \in R\text{-}\mathsf{Mod}$ be a finitely generated torsion module. The Fundamental Theorem says that there exist (unique) ideals $(1) \neq (f_1) \supseteq (f_2) \supseteq \cdots \supseteq (f_d) \neq (0)$ such that $T \cong \oplus_i R/(f_i)$.

(a) Define the set $\mathrm{Ann}_R(T) := \{r \in R : \forall\, t \in T, rt = 0\} \subseteq R$. Prove that this is an ideal of R (called the annihilator ideal of the module).

Proof. Consider any $s_1, s_2 \in \mathrm{Ann}_R(T)$ and $r \in R$. Then for all $t \in T$ we have

$$(s_1 + rs_2)t = s_1t + rs_2t = 0 + r0 = 0,$$

and it follows that $s_1 + rs_2 \in \mathrm{Ann}_R(T)$ as desired. □

(b) Prove that $(f_d) \subseteq \mathrm{Ann}_R(T)$.

Proof. Consider any $r \in (f_d)$. Since $(f_1) \supseteq (f_2) \supseteq \cdots \supseteq (f_d)$ we have $r \in (f_i)$ for all $i \in \{1, \ldots, d\}$. Then for any $t = (s_1 + (f_1), \ldots, s_d + (f_d)) \in T$ we have

$$rt = (rs_1 + (f_1), \ldots, rs_d + (f_d)) = (0 + (f_1), \ldots, 0 + (f_d))$$

and it follows that $r \in \mathrm{Ann}_R(T)$. □

(c) Prove that $\mathrm{Ann}_R(T) \subseteq (f_d)$. [Hint: If $r \in \mathrm{Ann}_R(T)$ then, in particular, r annihilates the element $(1 + (f_1), \ldots, 1 + (f_d))$.]

Proof. Suppose that $r \in \mathrm{Ann}_R(T)$. Then in particular we have

$$(0 + (f_1), \ldots, 0 + (f_d)) = r(1 + (f_1), \ldots, 1 + (f_d)) = (r + (f_1), \ldots, r + (f_d)).$$

Since $r + (f_d) = 0 + (f_d)$ we conclude that $r \in (f_d)$. □

(d) Let K be a field and consider a matrix $A \in \mathrm{Mat}_n(K)$. Explain how this defines a $K[x]$-module structure on the vector space $V := K^n$.

The K-module structure on V is carried by a ring homomorphism

$$\lambda : K \to \mathrm{End}_{\mathsf{Ab}}(V)$$

and we want to extend this to a ring homomorphism $\lambda' : K[x] \to \mathrm{End}_{\mathsf{Ab}}(V)$. Since $\mathrm{im}\,\lambda \subseteq Z(\mathrm{End}_{\mathsf{Ab}}(V))$ we have a natural K-algebra structure on $\mathrm{End}_{\mathsf{Ab}}(V)$. Then since $K[x]$ is the free K-algebra there exists a unique such λ' sending $x \mapsto A$. ///

[Remark: For gory details see HW1 Problem 3(a).]

(e) Since the module V from part (d) is a finitely generated torsion $K[x]$-module and since $K[x]$ is a PID (don't prove either of these statements) we obtain a decomposition $V \cong \oplus_i K[x]/(f_i(x))$ for some unique non-constant monic polynomials $f_1(x)|f_2(x)|\cdots|f_d(x)$. Prove that $f_d(x)$ is the minimal polynomial of A. [Hint: Use (b) and (c).]

Proof. From (b) and (c) we know that $(f_d(x)) = \mathrm{Ann}_{K[x]}(V)$. On ther other hand,

$$\begin{aligned}
\mathrm{Ann}_{K[x]}(V) &= \left\{f(x) \in K[x] : \forall\, v \in V, \lambda_{f(x)}(v) = 0\right\} \\
&= \{f(x) \in K[x] : \forall\, v \in V, f(A)v = 0\} \\
&= \{f(x) \in K[x] : f(A) = 0\} \\
&= (m_A(x)).
\end{aligned}$$

Since $f_d(x)$ and $m_A(x)$ are both monic we conclude that $f_d(x) = m_A(x)$. □