**Math 761** **Fall 2015**
**Midterm Exam** **Drew Armstrong**

**Problem 1. Categories.** Let $\mathcal{C}$ be a category.

(a) Define what it means for two objects $X, Y \in \mathcal{C}$ to be isomorphic.

We say that $X, Y \in \mathcal{C}$ are isomorphic if there exist morphisms $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\alpha \circ \beta = \mathrm{id}_Y$ and $\beta \circ \alpha = \mathrm{id}_X$.

(b) Define initial objects in $\mathcal{C}$.

We say that $X \in \mathcal{C}$ is an initial object if for all objects $Y \in \mathcal{C}$ we have $|\mathrm{Hom}_{\mathcal{C}}(X, Y)| = 1$.

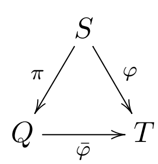
(c) Prove that any two initial objects $X, Y \in \mathcal{C}$ are isomorphic.

Let $X, Y \in \mathcal{C}$ be initial objects. By definition there exist (unique) morphisms $\alpha : X \to Y$ and $\beta : Y \to X$. Now consider the morphism $\alpha \circ \beta : Y \to Y$. Since $|\mathrm{Hom}_{\mathcal{C}}(Y, Y)| = 1$ we must have $\alpha \circ \beta = \mathrm{id}_Y$. Similarly, since $|\mathrm{Hom}_{\mathcal{C}}(X, X)| = 1$ we have $\beta \circ \alpha = \mathrm{id}_X$. We conclude that $X$ and $Y$ are isomorphic.

**Problem 2. Quotients.** Let $\sim$ be an equivalence relation on a set $S$.

(a) Define what it means for $\pi : S \to Q$ to be a $\sim$-quotient map.

We say that a function $\pi : S \to Q$ is a $\sim$-quotient map if
- For all $x, y \in S$ we have $(x \sim y) \Rightarrow (\pi(x) = \pi(y))$.
- Given a function $\varphi : S \to T$ satisfying $(x \sim y) \Rightarrow (\varphi(x) = \varphi(y))$ for all $x, y \in S$, there exists a unique function $\bar{\varphi} : Q \to T$ such that the following diagram commutes:

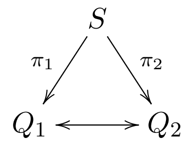



(b) Prove that a $\sim$-quotient map exists and say in what sense it is unique.

Given $x \in S$ we define the equivalence class $[x] := \{y \in S : x \sim y\}$. Now consider the set of equivalence classes $S/\!\sim \; := \{[x] : x \in S\}$. Since $(x \sim y) \Rightarrow ([x] = [y])$, the prescription $\pi(x) := [x]$ determines a well-defined function $\pi : S \to S/\!\sim$ satisfying the first property of a $\sim$-quotient map.

To establish the second property, let $\varphi : S \to T$ be any function satisfying $(x \sim y) \Rightarrow (\varphi(x) = \varphi(y))$. If there exists a function $\bar{\varphi} : S/\!\sim \; \to T$ satisfying the commutative diagram it must satisfy the prescription $\bar{\varphi}([x]) = \varphi(x)$ for all $x \in S$. Then since $([x] = [y]) \Rightarrow (x \sim y) \Rightarrow (\varphi(x) = \varphi(y))$, this prescription **does** define a function. ///

A quotient map is unique in the following sense: Let $\pi_1 : S \to Q_1$ and $\pi_2 : S \to Q_2$ be two $\sim$-quotient maps. Then there exists a unique bijection $Q_1 \longleftrightarrow Q_2$ such that the following diagram commutes:

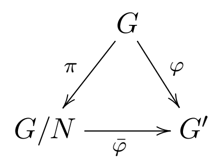



The uniqueness follows from Problem 1(c).

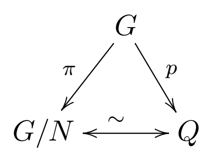
**Problem 3. First Isomorphism Theorem.** Let $N \trianglelefteq G$ be a normal subgroup.

(a) Define the universal property of a group quotient $\pi : G \to G/N$ and say in what sense a quotient is unique.

If $\varphi : G \to G'$ is any group homomorphism such that $N \subseteq \ker \varphi$, then there exists a unique group homomorphism $\bar{\varphi} : G/N \to G'$ such that the following diagram commutes:

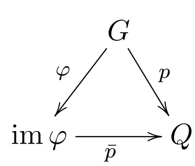



If $p : G \to Q$ is any other "$N$-quotient" satisfying this universal property then there exists a unique group isomorphism $G/N \overset{\sim}{\longleftrightarrow} Q$ such that the following diagram commutes:




(b) Now let $\varphi : G \to G'$ be a group homomorphism. Use the universal property from part (a) to prove that $G/\ker\varphi \approx \operatorname{im}\varphi$. [Hint: You can assume that the quotient $\pi : G \to G/\ker\varphi$ from part (a) exists.]

Since $\ker\varphi \trianglelefteq G$ we have the standard quotient map $\pi : G \to G/\ker\varphi$. I claim that the homomorphism $\varphi : G \to \operatorname{im}\varphi$ is another "$\ker\varphi$-quotient" map. Indeed, if $p : G \to G'$ is any group homomorphism such that $\ker\varphi \subseteq \ker p$ then any homomorphism $\bar{p} : \operatorname{im}\varphi \to Q$ such that




must satisfy the prescription $\bar{p}(\varphi(g)) = p(g)$ for all $g \in G$. Certainly this $\bar{p}$ will be a homomorphism if it is well-defined, and it **is** well-defined because for all $g, h \in G$ we have

$$(\varphi(g) = \varphi(h)) \Rightarrow (gh^{-1} \in \ker\varphi) \Rightarrow (gh^{-1} \in \ker p) \Rightarrow (p(g) = p(h)).$$

Now the isomorphism $G/\ker\varphi \approx \operatorname{im}\varphi$ follows from the uniqueness of quotients. ///

**Problem 4. Group Products.** Consider a group $G$ with subgroups $H, K \subseteq G$.

(a) Prove that $HK := \{hk : h \in H, k \in K\}$ is a subgroup of $G$ if and only if $HK = KH$.

First assume that we have $HK = KH$. To show that $HK$ is a subgroup consider any two elements $h_1k_1, h_2k_2 \in HK$. Since $k_1k_2^{-1}h_2^{-1} \in KH \subseteq HK$, there exist $h \in H$ and $k \in K$ such that $k_1k_2^{-1}h_2^{-1} = hk$. Then we have

$$(h_1k_1)(h_2k_2)^{-1} = h_1(k_1k_2^{-1}h_2^{-1}) = h_1hk \in HK,$$

as desired.

Conversely, assume that $HK \subseteq G$ is a subgroup. To prove that $HK = KH$, first consider an element $hk \in HK$. Since $HK$ is a group there exists $h'k' \in HK$ such that $hkh'k' = 1$, hence $hk = (k')^{-1}(h')^{-1} \in KH$ as desired. Next, consider any element $kh \in KH$. Since $k = 1k \in HK$ and $h = h1 \in HK$ and since $HK$ is a subgroup we obtain $kh \in HK$ as desired. ///

(b) Prove that the multiplication map $\mu : H \times K \to HK$ is injective if and only if $H \cap K = 1$.

First assume that multiplication $\mu : H \times K \to HK$ is injective and consider any element $g \in H \cap K$. Note that $g \in H$ and $g^{-1} \in K$ so we can apply multiplication to get $\mu(g, g^{-1}) = gg^{-1} = 1$. But we also have $\mu(1, 1) = 1$, so injectivity of $\mu$ implies that $(1, 1) = (g, g^{-1})$, hence $g = 1$.

Conversely, assume that $H \cap K = 1$ and suppose that $\mu(h_1, k_1) = \mu(h_2, k_2)$ (i.e., $h_1k_1 = h_2k_2$) for some $(h_1, k_1), (h_2, k_2) \in H \times K$. Then we have

$$h_1k_1 = h_2k_2 \quad \Longrightarrow \quad h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K.$$

Since $H \cap K = 1$ this implies that $h_2^{-1}h_1 = k_2k_1^{-1} = 1$, hence $h_1 = h_2$ and $k_1 = k_2$. It follows that $(h_1, k_1) = (h_2, k_2)$ and we conclude that $\mu$ is injective. ///

**Problem 5. Short Exact Sequences.** Let $K$ be a field and consider the following short exact sequence of groups:

$$\mathbf{1} \longrightarrow \mathrm{SL}_n(K) \xrightarrow{i} \mathrm{GL}_n(K) \xrightarrow{\det} K^\times \longrightarrow \mathbf{1}.$$

(a) Find an explicit section of the determinant map $\mathrm{GL}_n(K) \to K^\times$ and conclude that $\mathrm{GL}_n(K) \approx \mathrm{SL}_n(K) \rtimes K^\times$.

Given $\alpha \in K^\times$ we will define the matrix

$$s(\alpha) := \begin{pmatrix} \alpha & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Note that for all $\alpha, \beta \in K^\times$ we have $s(\alpha)s(\beta) = s(\alpha\beta)$ and $\det(s(\alpha)) = \alpha$, so that $s : K^\times \to \mathrm{GL}_n(K)$ is a section. We conclude from the splitting lemma on HW3 that

$$\mathrm{GL}_n(K) \approx \mathrm{SL}_n(K) \rtimes K^\times.$$

(b) Now assume that $K = \mathbb{R}$ and $n$ is odd. In this case find an explicit retraction of the inclusion map $\mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ and conclude that $\mathrm{GL}_n(\mathbb{R}) \approx \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^\times$. [Hint: Since $n$ is odd, every $\alpha \in \mathbb{R}^\times$ has an obvious $n$-th root.]

First note that since $n$ is odd, every $\alpha \in \mathbb{R}^\times$ has a **unique** $n$-th root in $\mathbb{R}^\times$. This defines a **function** $\sqrt[n]{\cdot} : \mathbb{R}^\times \to \mathbb{R}^\times$. Then since the product of $n$-th roots is an $n$-th root, uniqueness implies that $\sqrt[n]{\cdot}$ is a **group homomorphism**. [For a general field $K$ and general $n$ this is not possible.]

Now given an invertible matrix $A \in \mathrm{GL}_n(\mathbb{R})$ we will define

$$r(A) := \frac{1}{\sqrt[n]{\det(A)}} \cdot A.$$

Since $A$ is an $n \times n$ matrix we have

$$\begin{aligned}
\det(r(A)) &= \det\left(\frac{1}{\sqrt[n]{\det(A)}} \cdot A\right) \\
&= \left(\frac{1}{\sqrt[n]{\det(A)}}\right)^n \det(A) \\
&= \frac{1}{\det(A)} \cdot \det(A) \\
&= 1,
\end{aligned}$$

hence $r(A) \in \mathrm{SL}_n(\mathbb{R})$. Since $\sqrt[n]{\cdot}$ is a function we obtain a function $r : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$. Then since $\sqrt[n]{\cdot}$ is a homomorphism we have

$$\begin{aligned}
r(A)r(B) &= \frac{1}{\sqrt[n]{\det(A)}} \cdot A \cdot \frac{1}{\sqrt[n]{\det(B)}} \cdot B \\
&= \frac{1}{\sqrt[n]{\det(A)}} \cdot \frac{1}{\sqrt[n]{\det(B)}} \cdot AB \\
&= \frac{1}{\sqrt[n]{\det(A)\det(B)}} \cdot AB \\
&= \frac{1}{\sqrt[n]{\det(AB)}} \cdot AB \\
&= r(AB),
\end{aligned}$$

for all $A, B \in \mathrm{GL}_n(\mathbb{R})$, hence $r$ is a homomorphism. Finally, since $r(i(A)) = r(A) = A$ for all $A \in \mathrm{SL}_n(\mathbb{R})$ we conclude that $r$ is a retraction of the inclusion map $i : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$. It follows from the splitting lemma proved in class that

$$\mathrm{GL}_n(\mathbb{R}) \approx \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^\times.$$

[Remark: This is a pretty special isomorphism. I asked MathOverflow for a topological or geometric interpretation but I didn't get one yet.]