Math 761 **Fall 2015**
Homework 3 **Drew Armstrong**

Problem 1. Modularity. Let $(\mathscr{L}, \leq, \wedge, \vee, 0, 1)$ be a lattice. For all $x, y \in \mathscr{L}$ we define the closed interval $[x, y] := \{z \in \mathscr{L} : x \leq z \leq y\}$.

(a) Prove that for all $a, b \in \mathscr{L}$ we have a Galois connection

$$a \vee (-) : [0, b] \rightleftarrows [a, 1] : (-) \wedge b.$$

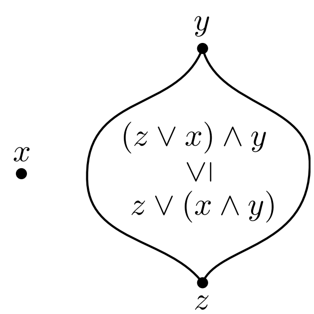
In other words, show that for all $x \in [0, b]$ and $y \in [a, 1]$ we have

$$x \leq (y \wedge b) \iff (a \vee x) \leq y.$$

(b) Given elements $x, y, z \in \mathscr{L}$ with $z \leq y$, there are two possible ways to map the element x into the interval $[z, y]$: by meeting with y and then joining with z, or by joining with z and then meeting with y. Prove that these two images are related by

$$z \vee (x \wedge y) \leq (z \vee x) \wedge y \tag{1}$$

as in the following picture:



We will say that $(a, b) \in \mathscr{L}^2$ is a modular pair if for all $x \leq b$ and $a \leq y$ the inequality (1) becomes an **equality**; that is, if we have

$$x \vee (a \wedge b) = (x \vee a) \wedge b, \quad \text{and} \tag{2}$$

$$a \vee (b \wedge y) = (a \vee b) \wedge y. \tag{3}$$

We will say that $a \in \mathscr{L}$ is a modular element if (a, b) is a modular pair for all $b \in \mathscr{L}$.

(c) If (a, b) is a modular pair, prove that the Galois connection from part (a) restricts to an isomorphism of lattices

$$[a \wedge b, b] \approx [a, a \vee b].$$

Problem 2. Normal $\Rightarrow$ Modular. Let G be a group and consider its lattice $\mathscr{L}(G)$ of subgroups. Let $H, N \in \mathscr{L}(G)$ with $N \trianglelefteq G$.

(a) Prove that N is a modular element of the lattice $\mathscr{L}(G)$ and conclude from Problem 1 that we have an isomorphism of lattices

$$[H \wedge N, H] \approx [N, H \vee N].$$

(b) Prove that the lattice isomorphism from part (a) lifts to an isomorphism of groups

$$\frac{H}{H \wedge N} \approx \frac{H \vee N}{N}.$$

[Hint: Since $N \trianglelefteq G$ we have $H \vee N = HN$ and $N \trianglelefteq HN$. Consider the function $\varphi : H \to HN/N$ defined by $\varphi(h) = (h1)N$.]

Problem 3. Modular $\not\Rightarrow$ Normal. Consider the dihedral group D_6 and the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Prove that we have an isomorphism of lattices

$$\mathscr{L}(D_6) \approx \mathscr{L}(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}).$$

Conclude that a modular element of the subgroup lattice is not necessarily normal.

Problem 4. A Zappa–Szép Product. Let $H, K \subseteq G$ be subgroups. We say that G is a Zappa–Szép product of H and K (and we write $G = H \bowtie K$) if $H \wedge K = 1$, $H \vee K = G$, and **neither** of H or K is normal in G.

(a) Let $H, K \subseteq G$ be finite subgroups, at least one of which is normal in G. Prove that

$$|H| \cdot |K| = |HK| \cdot |H \cap K|.$$

[Hint: Use Problem 2(b).]

(b) Prove that the result of part (a) holds even in the case when both of H and K are non-normal. [Hint: Let H act by left multiplcation on the set of left cosets G/K. Show that HK is the disjoint union of cosets in the orbit of $K \in G/K$. How many such cosets are there?]

(c) Consider a cycle $c = (i_1 i_2 \cdots i_k) \in S_n$ and a permutation $\pi \in S_n$. Prove that

$$\pi c \pi^{-1} = (\pi(i_1)\pi(i_2) \cdots \pi(i_k)).$$

Use this fact to describe the conjugacy classes of S_n.

(d) Let $G = S_4$, $H = \langle (1234), (12)(34) \rangle$, and $K = \langle (123) \rangle$. Prove that $G = H \bowtie K$. [Hint: Show that $H \approx D_8$ and $K \approx \mathbb{Z}/3\mathbb{Z}$. Now use parts (b) and (c).]

Problem 5. Right-Split Exact Sequences. A short exact sequence in the category of groups is a sequence of groups and homomorphisms of the form

$$\mathbf{1} \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow \mathbf{1}$$

that satisfies $\ker \alpha = 1$, $\operatorname{im} \alpha = \ker \beta$, and $\operatorname{im} \beta = H$. Given such a sequence, prove that the following two conditions are equivalent.

(1) There exists a group homomorphism $s : H \to G$ such that $\beta \circ s = \mathrm{id}_H$. [This s is called a section of β.] In this case we say that the short exact sequence is right-split.

(2) There exists a homomorphism $\varphi : H \to \mathrm{Aut}(N)$ and an isomorphism $\gamma : N \rtimes_\varphi H \to G$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & N & \longrightarrow & N \rtimes_\varphi H & \longrightarrow & H & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}_H} & & \\
\mathbf{1} & \longrightarrow & N & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \longrightarrow & \mathbf{1}
\end{array}$$

The maps in the top row are the obvious ones.

[Hint: To prove that (1) $\Rightarrow$ (2), consider any $h \in H$ and $n \in N$. Prove that there exists a unique $n' \in N$ such that $s(h)\alpha(n)s(h^{-1}) = \alpha(n')$. Call it $\varphi_h(n) := n'$. Show that this defines a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$. Now define a function $\gamma : N \rtimes_\varphi H \to G$ by $\gamma(n, h) := \alpha(n)s(h)$ and show that this is an isomorphism. To prove that (2) $\Rightarrow$ (1), define a function $s : H \to G$ by $s(h) := \gamma(1, h)$ and show that it has the desired properties.]