Math 761 **Fall 2015**
Homework 2 **Drew Armstrong**

Problem 1. Image and Preimage. Let $\varphi : G \to H$ be a group homomorphism and consider the Galois connection $\varphi : \mathscr{L}(G) \rightleftarrows \mathscr{L}(H) : \varphi^{-1}$ between image and preimage. Prove that for all subgroups $A \in \mathscr{L}(G)$ and $B \in \mathscr{L}(H)$ we have

- $\varphi^{-1}(\varphi(A)) = A \vee \ker \varphi$
- $\varphi(\varphi^{-1}(B)) = B \wedge \operatorname{im} \varphi$

Problem 2. Terminal Objects. Consider an object X in a category $\mathcal{C}$. We say that X is an initial object if for all objects Y we have $|\mathrm{Hom}_{\mathcal{C}}(X, Y)| = 1$, and we say that X is a final object if for all objects Y we have $|\mathrm{Hom}_{\mathcal{C}}(Y, X)| = 1$.

(a) Prove that any two initial objects (resp. final objects) are isomorphic in $\mathcal{C}$.
(b) Determine the initial and final objects in the category of sets.

Problem 3. Zero Objects and Zero Arrows. An object X in a category $\mathcal{C}$ is called a zero object if it is both initial and final. Suppose that the category $\mathcal{C}$ has a zero object 0 (which is unique up to isomorphism by Problem 1). Then between any two objects X and Y there is a unique zero arrow $0 : X \to Y$ defined by

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "0"] & 0 \arrow[r] & Y \end{tikzcd}$$

(a) Give an exmple of a category with no zero object.
(b) Describe the zero object and the zero arrows in the category of groups.

Problem 4. Universal Property of Kernels. Let $\mathcal{C}$ be a category with a zero object 0 and consider any arrow $G \xrightarrow{\varphi} G'$. Define a category $\mathcal{C}_\varphi$ whose objects are pairs (K, α) satisfying the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"'] \arrow[rr, bend left, "0"] & G \arrow[r, "\varphi"'] & G' \end{tikzcd}$$

and whose morphisms $(K_1, \alpha_1) \xrightarrow{\sigma} (K_2, \alpha_2)$ are arrows $K_1 \xrightarrow{\sigma} K_2$ in $\mathcal{C}$ satisfying

$$\begin{tikzcd} K_1 \arrow[dd, "\sigma"'] \arrow[dr, "\alpha_1"] \arrow[drr, bend left, "0"] & & \\ & G \arrow[r, "\varphi"'] & G' \\ K_2 \arrow[ur, "\alpha_2"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

If this category has a **final object** (K, α) we will call it the kernel of $G \xrightarrow{\varphi} G'$. (Note that the kernel consists of both an object K and an arrow $K \xrightarrow{\alpha} G$.)

(a) Verify that $\mathcal{C}_\varphi$ is a category.
(b) Prove that every homomorphism in the category of groups has a kernel. [Hint: You already know what the kernel "should" be.]

Problem 5. Universal Property of Products. Let $\mathcal{C}$ be a category. Given two objects A and B in $\mathcal{C}$ we define a new category $\mathcal{C}_{A,B}$ whose objects are triples (P, f, g) of the form

$$\begin{tikzcd} & A \\ P \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

and whose morphisms $(P_1, f_1, g_1) \xrightarrow{\sigma} (P_2, f_2, g_2)$ are arrows $P_1 \xrightarrow{\sigma} P_2$ in $\mathcal{C}$ satisfying

$$\begin{tikzcd} & & A \\ P_1 \arrow[r, "\sigma"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] & P_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\ & & B \end{tikzcd}$$

If this category has a **final object** (P, f, g) we will call it the product of A and B. (Note that the product consists of both the object P and the arrows f, g.)

(a) Verify that $\mathcal{C}_{A,B}$ is a category.
(b) Prove that products exist in the category of groups. [Hint: You already know what the product "should" be.]

Problem 6. Semi-Direct Products. Consider two groups N and G and a group homomorphism $\varphi : G \to \mathrm{Aut}_{\mathsf{Grp}}(N)$. We use φ to define a binary operation on the Cartesian product set $N \times G$ as follows:

$$(n_1, g_1) \bullet (n_2, g_2) := (n_1 \varphi_{g_1}(n_2), g_1 g_2).$$

Let $N \rtimes_\varphi G$ denote the triple $(N \times G, \bullet, (1_N, 1_G))$. We call this the semi-direct product of N and G with respect to φ.

(a) Prove that $N \rtimes_\varphi G$ is a group.
(b) Identify N and G with subgroups of $N \rtimes_\varphi G$ via the maps $n \mapsto (n, 1_G)$ for $n \in N$ and and $g \mapsto (1_N, g)$ for $g \in G$. Prove that

$$N \cap G = 1, \quad N \trianglelefteq N \rtimes_\varphi G, \quad \text{and} \quad NG = N \rtimes_\varphi G.$$

(c) Finally, prove that for all $n \in N$ and $g \in G$ we have $\varphi_g(n) = gng^{-1}$.

Problem 7. Dihedral Groups. A dihedral group is the semi-direct product of a cyclic group $\langle R \rangle$ of arbitrary order with a cyclic group $\langle F \rangle$ of order two via the homomorphism $\varphi : \langle F \rangle \to \mathrm{Aut}_{\mathsf{Grp}}(\langle R \rangle)$ defined by $\varphi_F(R) = R^{-1}$.

Now let G be a group containing two involutions $a, b \in G$ (i.e., $a, b \neq 1$ and $a^2, b^2 = 1$). Prove that the subgroup $\langle a, b \rangle \subseteq G$ generated by a and b is isomorphic to a dihedral group. [Hint: Let $F = a$ and $R = ab$.]