Math 761 **Fall 2015**
Homework 1 **Drew Armstrong**

On this homework you will further explore the idea of Galois connections. We will begin by defining a notion of Galois connection for general posets. Let $(P, \leq)$ and $(Q, \leq)$ be posets. A pair of maps $* : P \rightleftarrows Q : *$ is called a Galois connection if it satisfies the following property:

$$\boxed{\text{for all } p \in P \text{ and } q \in Q \text{ we have } p \leq q^* \Longleftrightarrow q \leq p^*}$$

Problem 1. Equivalent Definition. Prove that a pair of maps $* : P \rightleftarrows Q : *$ is a Galois connection (as defined above) if and only if the following two statements hold:

- For all $p \in P$ and $q \in Q$ we have
$$p \leq p^{**} \quad \text{and} \quad q \leq q^{**}.$$
- For all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have
$$p_1 \leq p_2 \Longrightarrow p_2^* \leq p_1^* \quad \text{and} \quad q_1 \leq q_2 \Longrightarrow q_2^* \leq q_1^*.$$

[Hint: Since the statements come in dual pairs, you only have to prove half of them.]

Recall that a lattice is a poset $(P, \leq)$ in which every pair of elements $x, y \in P$ has a (necessarily unique) join $x \vee y$ and meet $x \wedge y$. By induction, any **finite** subset $A \subseteq P$ also has a join $\bigvee A \in P$ and meet $\bigwedge A \in P$.

Problem 2. Lattice Structure. Let $* : P \rightleftarrows Q : *$ be a Galois connection. If, in addition, P and Q happen to be **lattices**, prove that for all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have

- $p_1^* \vee p_2^* \leq (p_1 \wedge p_2)^*$ and $q_1^* \vee q_2^* \leq (q_1 \wedge q_2)^*$
- $p_1^* \wedge p_2^* = (p_1 \vee p_2)^*$ and $q_1^* \wedge q_2^* = (q_1 \vee q_2)^*$

In the next problem you will show that the first inequalities are sometimes strict.

Problem 3. Counterexample. Consider the usual topology on the set of real numbers $\mathbb{R}$. Let $\mathscr{O} \subseteq 2^{\mathbb{R}}$ be the collection of open sets and let $\mathscr{C} \subseteq 2^{\mathbb{R}}$ be the collection of closed sets. Let $- : 2^{\mathbb{R}} \to 2^{\mathbb{R}}$ be the "topological closure" and let $\circ : 2^{\mathbb{R}} \to 2^{\mathbb{R}}$ be the "topological interior". One can check (you don't need to) that for all $O \in \mathscr{O}$ and $C \in \mathscr{C}$ we have

$$O \subseteq C^\circ \Longleftrightarrow O^- \subseteq C.$$

In other words, we have a Galois connection $- : \mathscr{O} \rightleftarrows \mathscr{C} : \circ$ where $\mathscr{O}$ is partially ordered by inclusion (" $\leq$ " = " $\subseteq$ ") and $\mathscr{C}$ is partially ordered by **reverse-inclusion** ("$\leq$ " = " $\supseteq$ "). Note that $\mathscr{O}$ is a lattice with $\wedge = \cap$ and $\vee = \cup$, whereas $\mathscr{C}$ is a lattice with $\wedge = \cup$ and $\vee = \cap$.

In this case, find **specific elements** $O_1, O_2 \in \mathscr{O}$ and $C_1, C_2 \in \mathscr{C}$ such that

$$O_1^- \vee O_2^- \lneq (O_1 \wedge O_2)^- \quad \text{and} \quad C_1^\circ \vee C_2^\circ \lneq (C_1 \wedge C_2)^\circ.$$

Now you will investigate under what conditions the first inequalities in Problem 2 become equalities.

Problem 4. Closed Elements. Let $* : P \rightleftarrows Q : *$ be a Galois connection between lattices P and Q. We will say that $p \in P$ (resp. $q \in Q$) is $**$-closed if $p^{**} = p$ (resp. $q^{**} = q$).

(a) Prove that the meet of any two $**$-closed elements is $**$-closed.
(b) Prove that the following two conditions are equivalent:
- The join of any two $**$-closed elements is $**$-closed.

- For all $**$-closed elements $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have
$$p_1^* \vee p_2^* = (p_1 \wedge p_2)^* \quad \text{and} \quad q_1^* \vee q_2^* = (q_1 \wedge q_2)^*.$$

Finally, let's put everything together. Basically, if we have a Galois connection between lattices in which joins of closed elements are closed, then this restricts to an **isomorphism** on their sublattices of closed elements. If $(P, \leq)$ is a poset we'll use the notation P^{op} for the same set of elements with the **opposite** partial order (and hence with meets and joins switched).

Problem 5. Galois Correspondence. Let $* : P \rightleftarrows Q : *$ be a Galois connection between lattices P and Q. Denote the image of $* : P \to Q$ by $P^* \subseteq Q$ and denote the image of $* : Q \to P$ by $Q^* \subseteq P$. We will think of these as subposets with the induced partial order.

(a) Prove that $Q^* \subseteq P$ and $P^* \subseteq Q$ are precisely the subposets of $**$-closed elements.

(b) Prove that the restricted maps $* : Q^* \rightleftarrows P^* : *$ are an **isomorphism of posets**:
$$Q^* \approx (P^*)^{\text{op}}.$$

(c) If, in addition, the join of any two $**$-closed elements is $**$-closed, prove that $Q^* \subseteq P$ and $P^* \subseteq Q$ are **sublattices**, and that the isomorphism from (b) is an **isomorphism of lattices**.

Epilogue: You might ask whether the definition of Galois connection given above is more general than the one discussed in class. The answer is: "yes and no". The answer is "yes" in the sense that this definition applies to more general posets. However, if P and Q happen to be Boolean lattices then the answer is "no". I will define a Boolean lattice as the collection of subsets of a set U, partially ordered by inclusion. Note that the lattice operations are $\wedge = \cap$ and $\vee = \cup$.

Problem 6. Boolean Galois Connections. Let S and T be sets and consider the corresponding Boolean lattices $P = 2^S$ and $Q = 2^T$. For any relation $R \subseteq S \times T$ and for any subsets $A \subseteq S$ and $B \subseteq T$ we will define the sets $A^R \subseteq T$ and $B^R \subseteq S$ as follows:

- $A^R = \{t \in T : \forall a \in A, aRt\}$
- $B^R = \{s \in S : \forall b \in B, sRb\}$

In class we called this an "abstract Galois connection" and we showed that it has many nice properties. Now let $* : P \rightleftarrows Q : *$ be a Galois connection of posets in the sense defined above. Prove that **there exists a unique relation** $R \subseteq S \times T$ such that for all $A \subseteq S$ and $B \subseteq T$ we have
$$A^* = A^R \quad \text{and} \quad B^* = B^R.$$

[Hint: Consider the singleton subsets of S and T. You will need to use the fact that the power set 2^U is a complete lattice, i.e., it is possible to take the intersection and union of arbitrary collections of subsets.]

Remark: The theory of Galois connections between posets is a special case of the theory of adjoint functors between categories. Maybe I will say something about this later; maybe not.