HW #1 due now.

Final Exam Thurs. 2:00-4:30 pm

The Final Exam is not cumulative. It will cover the material discussed since the Midterm. Here are the topics.

1) Functors & Natural Transformations.

Let $\mathcal{C}$ & $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ consists of

- a function $F: \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$

- for each pair $X, Y \in \text{Obj}(\mathcal{C})$ a function

  $F: \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{D}(F(X), F(Y))$

satisfy the two rules

- $\forall X \in \mathcal{C}$, $F(\text{id}_X) = \text{id}_{F(X)}$

- $\forall \alpha: X \to Y$ & $\beta: Y \to Z$

  $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$
A contravariant functor \( F : C \to D \) is the same as a covariant functor \( F : C \to D^{\text{op}} \) or \( F : C^{\text{op}} \to D \). That is, for all \( \alpha : x \to y \) \& \( \beta : y \to z \) we have

\[
F(\beta \circ \alpha) = F(\alpha) \circ F(\beta).
\]

Now let \( F, G : C \to D \) be covariant functors. A natural transformation

\[
\Phi : F \to G
\]

assigns to each object \( X \in \mathcal{C} \) a morphism \( \Phi(X) : F(X) \to G(Y) \) such that for all objects \( X, Y \in \mathcal{C} \) and morphisms \( \alpha : X \to Y \), the following square commutes:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\Phi(X)} & G(X) \\
\downarrow F(\alpha) & & \downarrow G(\alpha) \\
F(Y) & \xrightarrow{\Phi(Y)} & G(Y)
\end{array}
\]

We say that \( \Phi \) is a natural isomorphism \( F \cong G \) if \( \Phi(X) \) is an isomorphism \( \forall X \in \mathcal{C} \).
(2) The Category of G-sets.

Let $C$ & $D$ be categories with $C$ small. Then we can define the functor category

$$D^C$$

whose objects are functors $F: C \to D$ and whose morphisms are natural transformations.

For example, let $G$ be a group thought of as a category with one object. Then a functor $F: G \to \text{Set}$ is called a "$G$-set".

A $G$-set $F: G \to \text{Set}$ consists of a set

$$X := F(*)$$

and a function

$$F: G = \text{Aut}_G(*) \to \text{End}_{\text{Set}}(X).$$

The axioms of a functor imply that this is actually a group homomorphism

$$F: G \to \text{Aut}_{\text{Set}}(X).$$
To be more concrete, a G-set consists of a set \( X \) and a function \( G \times X \rightarrow X \) written as \((g, x) \rightarrow g(x)\) satisfying the two axioms

\[
\begin{align*}
\forall x \in X, \quad 1_G(x) &= x \\
\forall x \in X \quad \forall g, h \in G, \quad g(h(x)) &= (gh)(x).
\end{align*}
\]

This is equivalent to the previous definition via the identification

\[ g(x) = F(g)(x). \]

Now let \( X \) and \( Y \) be two \( G \)-sets with the actions \( G \times X \rightarrow X \) and \( G \times Y \rightarrow Y \) written implicitly. Then a morphism of \( G \)-sets is just a function \( \Phi : X \rightarrow Y \) such that for all \( x \in X \) and \( g \in G \) we have

\[ \Phi(g(x)) = g(\Phi(x)). \]

Check that is the same as a natural transformation of functors \( G \rightarrow \text{Set} \).
3) Fundamental Theorem of G-sets.

Given a G-set X and an element $x \in X$ we define

$$\text{Orb}_G(x) := \{ y \in X : \exists g \in G, \ y = g(x) \}$$

$$\text{Stab}_G(x) := \{ g \in G : g(x) = x \}$$

A Theorem (FTGS):

(i) For each $x \in X$, $\text{Stab}_G(x) \leq G$ is a subgroup and we have an isomorphism of G-sets

$$\text{Orb}_G(x) \cong G/\text{Stab}_G(x)$$

$$g(x) \iff g \text{ Stab}_G(x)$$

(ii) Given two subgroups $H, K \leq G$ we have

$$G/H \cong G/K$$

if and only if $\exists g \in G$, $gHg^{-1} = K$. 
Know how to prove the FTGS. Part (i) is straightforward. The key to Part (ii) is the identity

\[ \text{Stab}_G(g(x)) = g \text{Stab}_G(x) g^{-1}. \]

Two Examples from HW4:

- Let \( \text{Gr}_1(r,n) \) be the set of \( r \)-element subsets of \( \{1,2,\ldots,n\} \). Then we have an isomorphism of \( S_n \)-sets

\[ \text{Gr}_1(r,n) \cong S_n / (S_r \times S_{n-r}). \]

- Let \( \text{Gr}_K(r,n) \) be the set of \( r \)-dimensional subspaces of \( K^n \). Then we have an isomorphism of \( \text{GL}_n(K) \)-sets

\[ \text{Gr}_K(r,n) \cong \frac{\text{GL}_n(K)}{\text{Mat}_{r,n-r}(K) \times (\text{GL}_r(K) \times \text{GL}_{n-r}(K))}. \]
General Example:

Let $H, K \leq G$ be subgroups. Then we define an action of $H \times K$ on $G$ by

$$(H \times K) \times G \rightarrow G$$

$$(h, k, g) \rightarrow hgk^{-1}$$

The orbits are called double cosets

$$\text{Orb}_{H \times K}(g) = \{HgK : g \in G\}$$

If $H$ & $K$ are finite, be able to prove that

$$|HgK| = \frac{|H| \cdot |K|}{|H \cap gKg^{-1}|}$$

Hint: Show that

$$HgK \leftrightarrow \text{Orb}_H(gK) \times K$$

and $\text{Stab}_H(gK) = H \cap gKg^{-1}$.
4) The Class Equation.

Let \( G \) act on itself by \( g(h) := ghg^{-1} \).

The orbits are called conjugacy classes.

\[ K_g(a) := \{ b \in G : \exists g \in G, \ b = gag^{-1} \} \]

and the stabilizers are called centralizers.

\[ Z_g(a) := \{ g \in G : gag^{-1} = a \} \]

The intersection of all centralizers is called the center of \( G \).

\[ Z(G) := \bigcap_{a \in G} Z_g(a) \]

If \( K_1, K_2, \ldots, K_n \) are the classes of \( G \) and \( Z_1, Z_2, \ldots, Z_n \) are the corresponding centralizers (up to isomorphism), use the FTGS to prove that we have an isomorphism of \( G \)-sets:

\[ G \cong Z(G) \sqcup \left( \bigsqcup_{Z_i \neq G} \right) \]

Hint: \( K_g(a) = \{ a \} \iff a \in Z(G) \).
If \( G \) is finite, we obtain
\[
|G| = |Z(G)| + \sum_{Z \neq G} \frac{|G|}{|Z|}
\]
which is called the class equation.

(5) Application: Sylow Theory.

Let \( |G| = p^m \) with \( p \) prime and \( p \) divides \( m \).

A Theorem (Sylow):

(i) For all \( 0 \leq \beta \leq \alpha \), \( \exists \) subgroup \( H \leq G \) with \( |H| = p^\beta \).

(ii) If \( H, K \leq G \) are subgroups with
\( |K| = p^\alpha \) & \( |H| = p^\beta \) \((\beta \leq \alpha)\), then
\( \exists g \in G \) such that \( gHg^{-1} \leq K \).

(iii) If \( n_p := \# \{ K \leq G : |K| = p^\alpha \} \) then

\( n_p \mid m \)
\( n_p = 1 \mod p \).
You do not need to memorize the proof but you should know what goes into it.

The hardest part is the following lemma.

Lemma: let $A$ be a finite abelian group and let $p$ be prime. If $p | |A|$ then $A$ has an element of order $p$.

[We proved this with a tricky argument. We'll see the correct proof next semester when we prove the Fundamental Theorem of Finitely Generated $\mathbb{Z}$-modules.]

The rest of the proof is more straightforward.

Proof of (c): If $p^2 | |Z:1|$ then we're done by induction. Otherwise, the class equation says that $p | |Z(G)|$, hence $Z(G)$ has an element $z \in Z(G)$ of order $p$ (by the lemma). By induction $G/Z(G)$ has $p$-subgroups of all orders and we can lift these up to $G$. 

$\square$
Proof of (ii): Let $H, K \leq G$ with $|K| = p^\alpha$ and $|H| = p^\beta$ ($\beta < \alpha$). Decompose $G$ into double cosets to get

$$|G| = \sum \frac{|H| \cdot |K|}{|K \cap g^{-1}Hg|}.$$ 

Show by contradiction that one of the denominators has size $p^\beta$, hence

$$g \cdot Hg^{-1} \leq K.$$ 

Proof of (iii): Let $G \in \text{Syl}_p(G)$ by conjugation and fix $K \in \text{Syl}_p(G)$.

- By (ii) and FTGS we know that

$$|\text{Syl}_p(G)| = |\text{Orb}_e(K)| = |G|/|N_e(K)|.$$ 

Since $K \leq N_e(K)$ we have $p^\alpha \mid |N_e(K)|$ and it follows that $|\text{Syl}_p(G)| \mid m$.

- Now let $K \in \text{Syl}_p(G)$ by conjugation, so

$$|\text{Syl}_p(G)| = \sum \frac{|K|}{|\text{Stab}_K(H_i)|}.$$
For some $H_i \in \text{Syl}_p(K)$, show that $\text{Stab}_K(H_i) = K \iff H_i = K$ and hence

$$|\text{Syl}_p(G)| = 1 + \sum_{H_i \neq K} \frac{|K|}{|\text{Stab}_K(H_i)|}$$

$$= 1 \mod p.$$

You should know how to apply Sylow to study groups of small order.

**Example:** Prove that there is no simple group of order 12.

**Proof:** Let $|G| = 12 = 2^2 \cdot 3$.

Let $N_2 = |\text{Syl}_2(G)|$ & $N_3 = |\text{Syl}_3(G)|$.

By Sylow (i) we have $N_3 \geq 1$. If $N_3 = 1$ then by Sylow (ii) we obtain a normal subgroup of size 3, so assume that $N_3 \geq 2$. Then by Sylow (iii) we have

$$N_3 = 1 \mod 3,$$ hence $N_3 \geq 4$. 
Since these subgroups intersect trivially (they are cyclic), G must contain at least 8 elements of order 3.

But then there is room for only one Sylow 2-subgroup (of size 4), which must therefore be normal.

6. Finite matrix groups.

You should also remember that

\[ |GL_n(q)| = q^{\binom{n}{2}} (q-1)^{n-1} [n]_q! \]

\[ |SL_n(q)| = q^{\binom{n}{2}} (q-1)^{n-1} [n]_q! \]

\[ |PGL_n(q)| = \frac{q^{\binom{n}{2}} (q-1)^{n-1} [n]_q!}{\gcd(n,q-1)} \]

In case I want to use these as an example somewhere.