Tues Feb 12

Continued...

Rep. Theory of finite groups developed very quickly.

Dedekind $\rightsquigarrow$ Frobenius $\rightsquigarrow$ Done
letter: July 8, 1896 1901.

Dedekind considered the "Group Determinant"

Eg The group $\mathbb{Z}/2\mathbb{Z}$ has group table

+	0	1
0	0	1
1	1	0

Replace group elts by commutative variables $0, 1 \rightsquigarrow x_0, x_1$ and take determinant:

$$\det \begin{pmatrix} x_0 & x_1 \\ x_1 & x_0 \end{pmatrix} = x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1)$$

It splits!

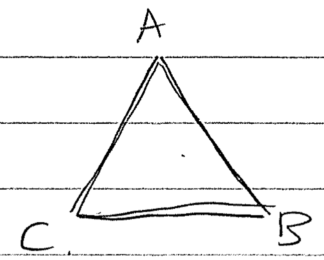
Dedekind observed/proved that the group det. splits over $\mathbb{C}$ for all abelian groups.

Eg $\mathbb{Z}/3\mathbb{Z} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{pmatrix}$

$\Longrightarrow \det = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3)$

where $\omega = e^{2\pi i/3}$.

But then he looked at dihedral D_6.



let $x_1 = id$
$x_2 = (ABC)$
$x_3 = (ACB)$
$x_4 = (BC)$
$x_5 = (AC)$
$x_6 = (AB)$

$$\Longrightarrow D_6 = \begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_3 & x_5 & x_6 & x_4 \\ x_3 & x_2 & x_1 & x_6 & x_4 & x_5 \\ x_4 & x_5 & x_6 & x_1 & x_3 & x_2 \\ x_5 & x_6 & x_4 & x_2 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_3 & x_2 & x_1 \end{pmatrix}$$

$$\Rightarrow \det = (u+v)(u-v)(u_1u_2 - v_1v_2)^2$$

where
$$u = x_1 + x_2 + x_3 \qquad v = x_4 + x_5 + x_6$$
$$u_1 = x_1 + \omega x_2 + \omega^2 x_3 \qquad v_2 = x_4 + \omega x_5 + \omega^2 x_6$$
$$u_2 = x_1 + \omega^2 x_2 + \omega x_3 \qquad v_3 = x_4 + \omega^2 x_5 + \omega x_6.$$

Dedekind didn't understand.
Frobenius understood. He only labeled the conjugacy classes.

$$x_1 = x, \quad x_2 = x_3 = y, \quad x_4 = x_5 = x_6 = z$$

Then we get

$$\det = (x+2y+3z)(x+2y-3z)\left((x-y)^2\right)^2$$

Taking degrees gives

$$|D_6| = 6 = 1 + 1 + 2^2$$

Let me explain.

Recall: A representation of finite G is a group hom

$$\rho : G \longrightarrow GL_n(\mathbb{C}).$$

The character of ρ is its trace.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, "\chi_\rho"'] & GL_n(\mathbb{C}) \arrow[r, "tr"] & \mathbb{C} \end{tikzcd}$$

not a hom.

Recall: $tr(AB) = tr(BA)$.

But $tr(AB) \neq tr(A)\, tr(B)$.

Still, χ_ρ is a class function

$$\begin{aligned} \chi_\rho(ghg^{-1}) &= tr(\rho(ghg^{-1})) \\ &= tr(\rho(g)\rho(h)\rho(g)^{-1}) \\ &= tr(\cancel{\rho(g)^{-1}\rho(g)}\, \rho(h)) \\ &= tr(\rho(h)) = \chi_\rho(h). \end{aligned}$$

Constant on conjugacy classes.

Class functions $G \to \mathbb{C}$ form a $\mathbb{C}$-vector space of dim = # conj classes.

We can even make a hermitian product.
Given class funcs. $\chi, \varphi : G \to \mathbb{C}$, let

$$\langle \chi, \varphi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\varphi(g)}$$

So what?
Characters live in here.

Eg $G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ has just 2 irr. reps.

$\rho_1(+1) = +1$ $\qquad$ $\rho_2(+1) = +1$
$\rho_1(-1) = +1$ $\qquad$ $\rho_2(-1) = -1$
"triv." $\qquad$ "the other one"

Since ρ_1, ρ_2 have dim 1, $\chi_{\rho_1} = \rho_1$, $\chi_{\rho_2} = \rho_2$

Class functions $\mathbb{Z}/2\mathbb{Z}$ have a standard orthonormal basis:

$e_1 : \begin{array}{l} \{1\} \to 1 \\ \{-1\} \quad 0 \end{array}$ $\qquad$ $e_2 : \begin{array}{l} \{1\} \to 0 \\ \{-1\} \quad 1 \end{array}$

Surpise: irr. chars. χ_1, χ_2 also form an orthonormal basis.

$$\langle \chi_1, \chi_1 \rangle = \frac{1}{2}\left(1 \cdot \bar{1} + 1 \cdot \bar{1}\right) = 1$$

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{2}\left(1 \cdot \bar{1} + 1 \cdot (\overline{-1})\right) = 0$$

$$\langle \chi_2, \chi_2 \rangle = \frac{1}{2}\left(1 \cdot \bar{1} + (-1)(\overline{-1})\right) = 1$$

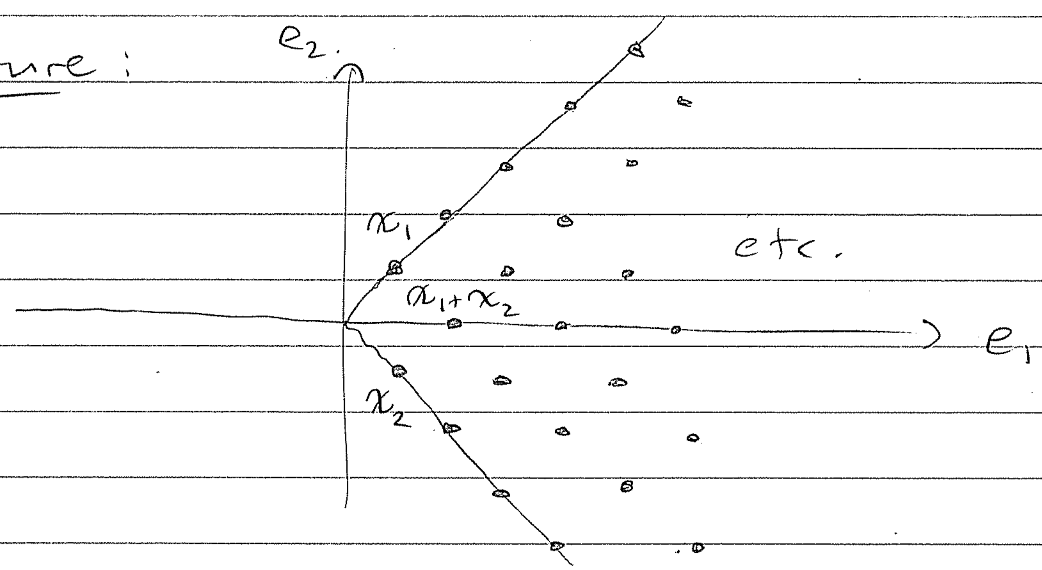
Characters are also closed under pointwise addition and multiplication.

E.g. $\chi_1 + \chi_1 : \begin{matrix}\{1\} \to 2 \\ \{-1\} \quad 0\end{matrix}$, $\quad \chi_1\chi_2 : \begin{matrix}\{1\} \to 1 \\ \{-1\} \quad -1\end{matrix}$.

$$\chi_1\chi_2 = \chi_2.$$

Proof: $\chi_{\rho_1} + \chi_{\rho_2} = \chi_{\rho_1 \oplus \rho_2}$.
$\chi_{\rho_1}\chi_{\rho_2} = \chi_{\rho_1 \otimes \rho_2}$ $\boxtimes$

Picture:



Theorem:

(1) The characters form a full-rank $\mathbb{Z}$-cone in the space of class funcs.

(2) The irreducible chars. form an orthonormal $\mathbb{Z}$-basis for the cone

(3) Hence, # irr. chars = # conj classes.

Problem (Generally Hard):

Say G has irr chars $\chi_1, \chi_2, \dots, \chi_N$.
Multiplication gives structure constants.

$$\chi_i \chi_j = \sum_{k=1}^{N} c_{ij}^{k} \chi_k.$$

for some $c_{ij}^{k} \in \mathbb{N}$.

Describe these numbers!

Eg The irr. chars of D_6 are

		classes		
		x	y	z
chars	χ_1	1	1	1
	χ_2	1	1	-1
	χ_3	2	-1	0

Theorem (Frobenius):

Let G have conj classes $G = C_1 \cup C_2 \cup \cdots \cup C_N$.
and irr. characters $\chi_1, \chi_2, \ldots, \chi_N$.
Then the group determinant (with $x_g = x_i$ for all $g \in C_i$) has $\mathbb{C}$-irreducible factors

$$\Phi_i = \frac{1}{|C_i|}\left(\chi_i(C_1)x_1 + \cdots + \chi_i(C_N)x_N\right).$$

appearing with multiplicity $(\dim \chi_i)^2$.

i.e.

$$\det = \Phi_1^{(\dim \chi_1)^2} \Phi_2^{(\dim \chi_2)^2} \cdots \Phi_N^{(\dim \chi_N)^2}.$$

Fun Corollary:

$$|G| = \sum_{i=1}^{N} (\dim \chi_i)^2$$

Example : D_6

	Classes			
	x	y	z	
sizes	1	2	3	
χ_1	1	1	1	$\longrightarrow \Phi_1 = x + 2y + 3z$
χ_2	1	1	-1	$\longrightarrow \Phi_2 = x + 2y - 3z$
χ_3	2	-1	0	$\longrightarrow \Phi_3 = \frac{1}{2}(2x - 2y)$

$$\Longrightarrow \det = (x+2y+3z)^{1^2}(x+2y-3z)^{1^2}(x-y)^{2^2}$$

///

Q: How strong is character theory ?
Does it determine the group ?

A: NO. For example,

$D_8 = D_{2,2,4}$ and

$Q_8 = D^*_{2,2,2}$

have the same character table

Let's compute it for $D_8 = \langle a^2 = b^2 = (ab)^4 = 1 \rangle$.

Linear (dim 1) characters.

Say $\chi : D_8 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$

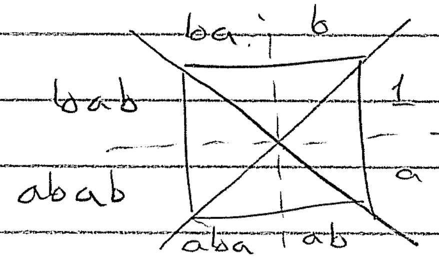
Then $\chi(a)^2 = \chi(a^2) = \chi(1) = 1$

$\chi(b)^2 = \chi(b^2) = \chi(1) = 1$.

$\Longrightarrow$ 4 choices $\chi(a) = \pm 1$

$\chi(b) = \pm 1$.

Conjugacy Classes:



1, rotate 90°, rotate 180°, reflect vertex, reflect edge

ab, ba   abab   b, aba   a, bab.

	1	ab, ba	abab	b, aba	a, bab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	-1	1	-1	1
χ_4	1	1	1	-1	-1

Any more? Yes there is one.

The defining representation

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$ab, ba = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$aba, bab = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$1, abab = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Char Table of D_8 :

	1,	$abab$,	ab ba,	b aba,	a bab
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

Claim: Also the char table of Q_8