Tues Feb 5

The "3-sphere"

$$S^3 = \{ \vec{x} \in \mathbb{R}^4 : \|\vec{x}\| = 1 \} \subseteq \mathbb{R}^4.$$

Recall: It's a group.

$S^3 = Sp(1)$ via

$$(x_0, x_1, x_2, x_3) \longleftrightarrow x_0 \hat{1} + x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$$

OR $S^3 = SU(2)$ via

$$(x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix}$$

Note $Sp(1) = SU(2)$ via

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\hat{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

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If we identify the algebra

$$\mathbb{R}^4 = \mathbb{H} = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

$$= \left\{ \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix} : x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

The imaginary quaternions $(x_0 = 0)$ form a hyperplane $su(2) = \mathbb{R}^3 \subseteq \mathbb{R}^4$, where

$$su(2) := \left\{ \begin{pmatrix} x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & -x_1 i \end{pmatrix} ; x_1, x_2, x_3 \right\}$$

$$= \left\{ A \in Mat_2(\mathbb{C}) : A^* = -A,\ tr(A) = 0 \right\}$$

= space of skew-hermitian trace-free 2×2 matrices.

Lemma: The dot product on $\mathbb{R}^3$ corresponds to

$$\langle u, v \rangle = -\frac{1}{2} tr(uv)$$

on $su(2)$.

<u>Proof</u>: Let $u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$
$v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$.

We have seen that

$$uv = -(u_1v_1 + u_2v_2 + u_3v_3)\,\hat{1}$$
$$+ \left[(u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}\right].$$

Since $tr(\hat{1}) = 2$ and $tr(\hat{i}) = tr(\hat{j}) = tr(\hat{k}) = 0$, we have.

$$\begin{aligned} tr(uv) &= -2(u_1v_1 + u_2v_2 + u_3v_3) \\ &= -2\langle u, v\rangle \end{aligned}$$

$\blacksquare$

Now observe that $SU(2)$ acts on $su(2)$ by conjugation

Indeed, given $p \in SU(2)$ define $\gamma_p : \mathbb{H} \to \mathbb{H}$ by $u \mapsto pup^*$.

① It's an isometry:

$$\begin{aligned} \|pup^* - pvp^*\| &= \|p(u-v)p^*\| \\ &= \cancel{\|p\|}\cdot\|u-v\|\cdot\cancel{\|p^*\|} \\ &= \|u-v\| \end{aligned}$$

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(2) It preserves scalar matrices:

$$\mathfrak{su}(2)^{\perp} = \left\{ x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : x_0 \in \mathbb{R} \right\}$$

since
$$\begin{aligned} P(x_0 I)P^* &= x_0 P I P^* \\ &= x_0 P P^* \\ &= x_0 I \end{aligned}$$
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(3) Therefore γ_P preserves the orthogonal complement $\mathfrak{su}(2)$.

[Remark: This is called the Adjoint action of $SU(2)$ on $\mathfrak{su}(2)$.

$SU(2)$ = Lie group
$\mathfrak{su}(2)$ = Lie algebra

Given $u, v \in \mathfrak{su}(2)$,

$$\langle u, v \rangle = -\frac{1}{2} \operatorname{tr}(uv)$$

is called the "Killing form".

$SU(2) \curvearrowright \mathfrak{su}(2)$ preserves the Killing form.]

Finally, we show that $SU(2) = S^3$ on $su(2) = \mathbb{R}^3$ by <u>rotations</u>.

<u>Theorem</u>: Given $P \in SU(2)$, the map $\gamma_P : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\gamma_P U = PUP^*$ is a rotation.

<u>Proof</u>: It's linear:

$$\gamma_P(U+V) = \gamma_P(U) + \gamma_P(V)$$
$$\gamma_P(rU) = r\gamma_P(U)$$

and we saw it's an isometry, hence $\gamma_P \in O(3)$.

We must show $\det \gamma_P = +1$.
Well we know $\det \gamma_P = \pm 1$.
Since

i) $\det : SU(2) \to \mathbb{C}$ is continuous
ii) $SU(2) = S^3$ is path-connected.
iii) $I \in SU(2)$ has $\det \gamma_I = +1$

we conclude $\det \gamma_P = +1 \ \forall \ P \in SU(2)$.

Finally, Euler (1775) says γ_P is a rotation $\boxtimes$

We get a group homomorphism

$$\begin{aligned} SU(2) &\longrightarrow SO(3) \\ P &\longmapsto \gamma_P \end{aligned}$$

called the "spin homomorphism".

Fact: The kernel is $\{\pm I\}$

Proof: If $\gamma_P = id$ then $PUP^{-1} = U$ (ie. $PU = UP$) for all $U \in su(2)$.
But every $Q \in SU(2)$ can be written in polar form

$$Q = cI + sU$$

where $U \in su(2)$, $\|U\| = 1$, $c^2 + s^2 = 1$.
ie.

$$Q = \cos\theta\, I + \sin\theta\, U \quad \text{for some } \theta \in \mathbb{R}.$$

Then P commutes with Q, so P is in the center $\{\pm I\}$ of $SU(2)$.

$\boxtimes$

Fact: The map $SU(2) \to SO(3)$ is surjective.

Proof: Given $P \in SU(2)$, write

$$P = \cos\theta\, I + \sin\theta\, U$$

where $U \in su(2) = \mathbb{R}^3$ with $\|U\| = 1$, we claim $\gamma_P : \mathbb{R}^3 \to \mathbb{R}^3$ is rotation around $U \in \mathbb{R}^3$ by angle 2θ.

By conjugation, enough to show for $U = \hat{i}$.

$$P = \cos\theta\, I + \sin\theta\, \hat{i} = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Then we have $\gamma_P \hat{j} = P \hat{j} P^*$

$$= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{2i\theta} \\ e^{-2i\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cos 2\theta + i \sin 2\theta \\ \cos 2\theta - i \sin 2\theta & 0 \end{pmatrix}$$

$$= \cos 2\theta \, \hat{j} + \sin 2\theta \, \hat{k}.$$

Hence γ_p acts on $\mathbb{R}\hat{j} + \mathbb{R}\hat{k}$ by rotating 2θ.

$\boxtimes$

Topologically, we have a "double cover"

$$SU(2) \xrightarrow{2:1} SO(3)$$

which implies

$$SO(3) = \frac{SU(2)}{\{\pm I\}} = \frac{S^3}{\text{antipodal points identified}}$$

$$= \mathbb{R}P^3$$

real projective 3-space.

The Exponential Map.

Given square matrix $A \in \mathrm{Mat}_n(\mathbb{C})$, define the matrix

$$e^{tA} := I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

It converges uniformly.

Example: For any $P = \cos\theta\, I + \sin\theta\, U$ in $SU(2)$ with $U \in su(2)$ with $\|U\| = 1$, we have

$$e^{\theta U} = \cos\theta\, I + \sin\theta\, U.$$

Conclusion: We get a surjective map.

$$\exp : su(2) \longrightarrow SU(2).$$
$$\underset{tr=0}{U} \longmapsto \underset{det=1}{e^{U}}.$$

Exercise: $\det(e^A) = e^{tr(A)}$

Q: Is $\exp : su(2) \to SU(2)$
a "homomorphism"?

Well, $su(2)$ is not a group!

$su(2)$ is a Lie algebra with "Lie bracket"

$$[u, v] := uv - vu$$

$$= (-\cancel{u \cdot v} + u \times v) - (-\cancel{v \cdot u} + \cancel{v \times u}^{\,-u\times v})$$

$$= 2(u \times v)$$

essentially cross product.

NOT Associative: Instead we have

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]],$$

which is the derivative of "associative".

That is, for all $u, v \in su(2)$
we have

$$\frac{d}{dt} e^{tu} v e^{-tu} \Big|_{t=0} = [u, v]$$

Or you can kind of get "associativity" from $[\,,\,]$ and the Killing form $\langle\,,\,\rangle$:

$$\langle [u,v], w \rangle = \langle u, [v,w] \rangle$$

Slogan:

Lie algebras generalize the dot and cross product in $\mathbb{R}^3 = su(2)$.

$$(u \times v) \cdot w = u \cdot (v \times w).$$