Tues Jan 29

The Poincaré Disk :

In each dimension, the geometries of constant (sectional) curvature $\mathcal{K}$ are classified as

① Elliptic/Spherical $(\mathcal{K} > 0)$

eg $S^n \subseteq \mathbb{R}^{n+1}$

Isometry groups : $SO(n+1) < O(n+1)$

② Euclidean/Affine $(\mathcal{K} = 0)$

eg $\mathbb{R}^n$

Isometry groups : $\mathrm{Isom}^+(\mathbb{R}^n) < \mathrm{Isom}(\mathbb{R}^n)$

③ Hyperbolic $(\mathcal{K} < 0)$

Isometry groups : $SO^+(n,1) < O^+(n,1)$

☹ $\nexists$ Euclidean embedding

so we have to be tricky . . .

The Poincare model for H^2:

$$H^2 = \{ z \in \mathbb{C} : |z| < 1 \}$$

(open unit disk).

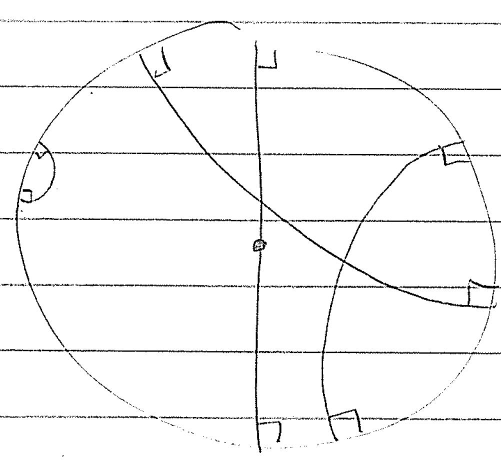
with metric tensor

$$ds^2 = \frac{dz\, d\bar{z}}{(1-|z|^2)^2}$$

hence the metric is

$$\text{dist}(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right| .$$

"Straight lines" (i.e. geodesics) are arcs of circles $\perp$ the boundary



Recall the abstract von Dyck groups:

$$D(p,q,r) = \langle X^p = Y^q = Z^r = XYZ = 1 \rangle$$

Theorem: $\forall\, p,q,r \geq 1$, $D(p,q,r)$ is $\approx$ discrete group of isometries of a 2D space of constant curvature.

(1) If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ then

$$D(p,q,r) < SO(3)$$

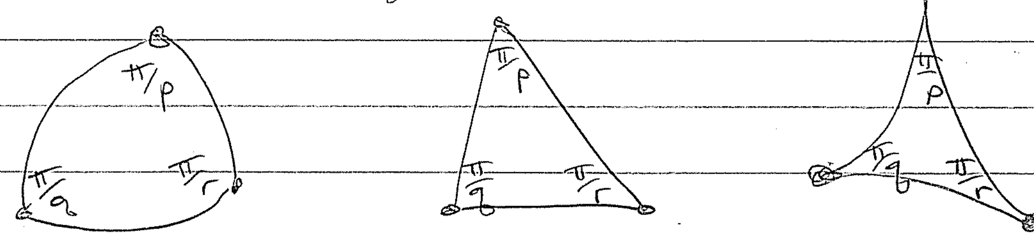
(2) If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ then

$$D(p,q,r) < Isom^+(\mathbb{R}^2)$$

(3) If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ then

$$D(p,q,r) < SO^+(2,1)$$

In all cases, X, Y, Z are rotations by $2\pi/p$, $2\pi/q$, $2\pi/r$ around the vertices of a triangle



Now weaken the definition

$$D^*(p,q,r) = \langle X^p = Y^q = Z^r = XYZ \rangle$$

Miracle: If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ then
$\exists\, \delta \in D^*(p,q,r)$ such that $\delta^2 = 1$ and

$$D^*(p,q,r) = \langle X^p = Y^q = Z^r = XYZ = \delta \rangle$$

think $\delta =$ "-1"

From this we get a surjective map

$$D^*(p,q,r) \twoheadrightarrow D(p,q,r)$$

with kernel $\langle \delta \rangle$ of order 2.
It follows that

$$|D^*(p,q,r)| = 2\,|D(p,q,r)| < \infty$$

Q: What are these groups?

The groups $D^*(2,2,a)$ are called <u>dicyclic</u>.

eg

$$D^*(2,2,2) = \langle X^2 = Y^2 = Z^2 = XYZ = \delta \rangle$$

where $\delta^2 = 1$

Reminds me of this:

$$\boxed{i^2 = j^2 = k^2 = ijk = -1} \quad (*)$$

W.R. Hamilton, Oct 16, 1843

The Quaternions are

$$\mathbb{H} := \{a + bi + cj + dk : a,b,c,d \in \mathbb{R}\} / (*)$$

Q: Why do they exist?

A: Because $\mathbb{C}$ exists.

We have an explicit representation

$$\mathbb{H} \hookrightarrow \mathrm{Mat}_2(\mathbb{C})$$

given by

$$\hat{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{i} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{j} \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \hat{k} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

OR

$$a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k} \mapsto \begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix}$$

$$\doteq \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

Note: $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

We <u>identify</u> $\mathbb{H}$ with this embedding.

Note. $\det \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \alpha\overline{\alpha} + \beta\overline{\beta} \in \mathbb{R}$

$$= a^2 + b^2 + c^2 + d^2$$

This suggests a definition.

Given $q = a + ib + cj + dk \in \mathbb{H}$, define its absolute value

$$|q|^2 := a^2 + b^2 + c^2 + d^2.$$

Theorem: $|q_1 q_2| = |q_1| \cdot |q_2|$.

Proof:
$$\begin{aligned} |q_1 q_2|^2 &= \det(q_1 q_2) \\ &= \det(q_1)\det(q_2) \\ &= |q_1|^2 \cdot |q_2|^2 \end{aligned}$$
$\blacksquare$

Theorem: Quaternions are invertible.

Proof: Given $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathbb{H}$ we have

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}^{-1} = \frac{1}{a^2+b^2+c^2+d^2} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha' & \beta' \\ -\bar{\beta}' & \bar{\alpha}' \end{pmatrix} \in \mathbb{H}.$$

$\blacksquare$

Explicitly:

$$(a1+bi+cj+dk)^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a1-bi-cj-dk)$$

Hmm...

Define the quaternion conjugate

$$\overline{a+bi+cj+dk} := a-bi-cj-dk$$

Theorem: $q\bar{q} = |q|^2$ $\blacksquare$

In terms of complex conjugation

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

then $\bar{q} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \beta & \alpha \end{pmatrix} = \overline{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}}^{t}$

conjugate transpose.

Corollary: The group of unit quaternions

$$S\mathbb{H} := \{ q \in \mathbb{H} : q\bar{q} = |q|^2 = 1 \}$$

is isomorphic to a subgroup of $SU(2)$.

Theorem: In fact, $S\mathbb{H} = SU(2)$

<u>Proof</u>: Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$,

So $$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Longrightarrow$ (α, β) and (γ, δ) are orthonormal with respect to the standard hermitian form

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$$

We conclude that

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = z \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} \quad \text{for some } z \in \mathbb{C}$$

Hence $$A = \begin{pmatrix} \alpha & \beta \\ -z\bar{\beta} & z\bar{\alpha} \end{pmatrix}$$

$$\Longrightarrow \det(A) = z\left(\cancelto{1}{\alpha\bar{\alpha} + \beta\bar{\beta}}\right) = z$$

$$\Longrightarrow z = 1.$$

$\blacksquare$

Summary: $\mathbb{H}$ is a normed, associative division $\mathbb{R}$-algebra.

[Theorem (Frobenius): There aren't many of these; just $\mathbb{R}, \mathbb{C}, \mathbb{H}$]

So we can kind of do linear algebra.

Consider $\mathbb{H}^n$ with "inner product"

$$\left\langle \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \right\rangle := \underbrace{p_1 \bar{q}_1 + \cdots + p_n \bar{q}_n}_{\text{quaternion conjugates}}$$

Define the symplectic group

$$Sp(n) = \{ A \in Mat_n(\mathbb{H}) : \langle A\vec{p}, A\vec{q} \rangle = \langle \vec{p}, \vec{q} \rangle \ \forall \vec{p}, \vec{q} \in \mathbb{H}^n \}$$

$$= \{ A \in Mat_n(\mathbb{H}) : A\bar{A}^t = 1 \}$$

We just saw that

$$Sp(1) \approx SU(2).$$

[Recall $U(1) \approx SO(2)$]
Euler

Different Perspectives :

$$\begin{array}{ccccc} \mathbb{H} & & \mathbb{C} & & \mathbb{R} \\ & & U(1) & \xrightarrow{\sim} & SO(2) \\ Sp(1) & \xrightarrow{\sim} & SU(2) & \longrightarrow & ? \end{array}$$

In other words,

Q: What is the Real interpretation of quaternions ?