Tues Jan 22

Webpage up.

Recall: Discrete subgroups of $O(2)$ are cyclic

$$C_a = \left\langle \begin{pmatrix} \cos(2\pi/a) & -\sin(2\pi/a) \\ \sin(2\pi/a) & \cos(2\pi/a) \end{pmatrix} \right\rangle$$

and dihedral

$$D_{2a} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos(\frac{2\pi}{a}) & -\sin(\frac{2\pi}{a}) \\ \sin(\frac{2\pi}{a}) & \cos(\frac{2\pi}{a}) \end{pmatrix} \right\rangle$$

$$= \langle r, \rho : r^2 = \rho^a = 1, r\rho r^{-1} = \rho^{-1} \rangle$$

$$= \langle \rho \rangle \rtimes \langle r \rangle$$

Alternatively, let $s = r$ and $t = r\rho$

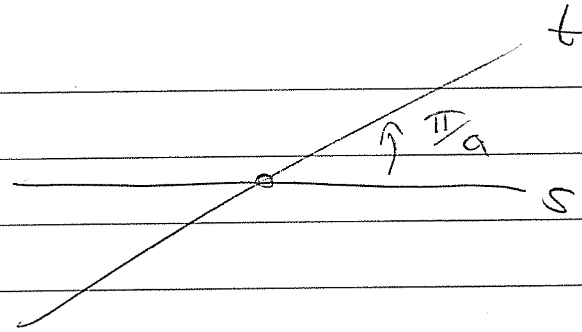
Then $\det(s) = \det(t) = -1$
"reflections"

$st = rr\rho = \rho$

$(st)^a = \rho^a = 1$

s and t have angle $\frac{2\pi}{a} \cdot \frac{1}{2} = \frac{\pi}{a}$.

Picture:

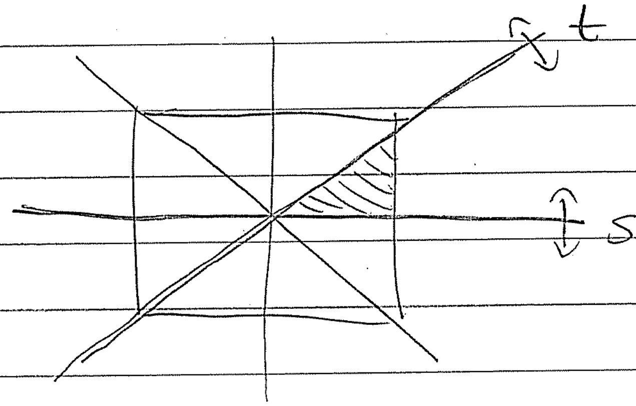


Just one relation: $(st)^a = 1$.

$D_{2a} = \langle s, t : s^2 = t^2 = 1,\ (st)^a = 1 \rangle$

Exercise: Verify this assertion.

Example: Symmetries of a square.

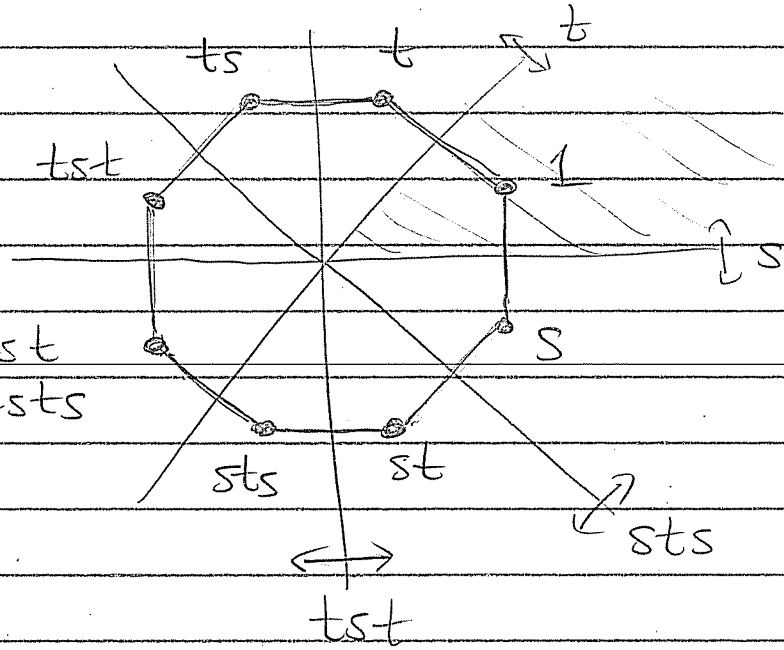


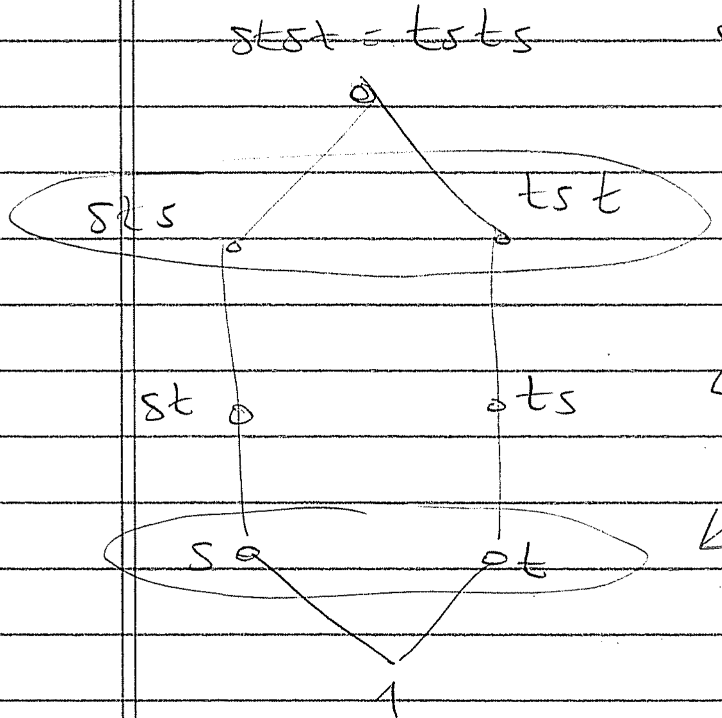
$D_8 = \langle s, t : s^2 = t^2 = 1,\ (st)^4 = 1 \rangle$

4 reflecting lines
divide the plane into 8 regions

$D_8 \curvearrowright$ regions simply-transitively
("relgularly").
("Torsor").

Choose a "basepoint" in the "fundamental region" and call it 1





rotations
det = +1

reflections
det = −1

"weak (Bruhat) order on D_8".

Define a length function on D_{2a}.

$\ell : D_{2a} = \langle s, t \rangle \longrightarrow \mathbb{N}$

Given $w \in D_{2a}$ let $\ell(w) := \min \{ k \in \mathbb{N} :$
$w =$ product of k "s"s and "t"s $\}$

Note :

- $\det(w) = (-1)^{\ell(w)}$
- $\exists$ unique "longest" element $w_0 \in D_{2a}$ with $\ell(w_0) = a$
- If a is even then

$$w_0 = (st)^{a/2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \text{the "antipodal" map.}$$

(rotation by 180°)

If a is odd then

$$w_0 = (st)^{\frac{a-1}{2}} s = \text{a reflection}$$

- D_{2a} contains
  - $a-1$ rotations
  - a reflections
  - 1 identity.

subgroup $D_{2a}^{+} = C_a$.

Next: $SO(3)$

Recall (Cartan–Dieudonné): Every nontrivial $A \in SO(3)$ has an axis of rotation; i.e. $\exists\ B \in SO(3)$ with

$$BAB^{-1} = \left(\begin{array}{c|cc} 1 & & \\ \hline & \cos\alpha & -\sin\alpha \\ & \sin\alpha & \cos\alpha \end{array}\right)$$

Q: Classify discrete (i.e. finite) subgroups of $SO(3)$.

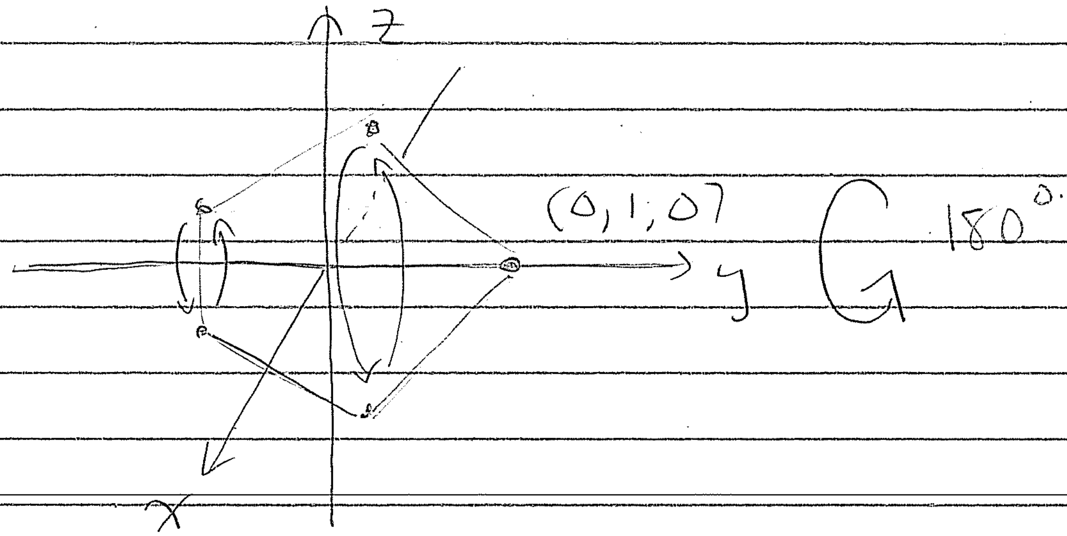
Two Infinite Families:

(1) Cyclic groups

$$C_a = \left\langle \left(\begin{array}{c|cc} 1 & & \\ \hline & \cos\left(\frac{2\pi}{a}\right) & -\sin\left(\frac{2\pi}{a}\right) \\ & \sin\left(\frac{2\pi}{a}\right) & \cos\left(\frac{2\pi}{a}\right) \end{array}\right) \right\rangle.$$

(2) Dihedral groups? Yes.

Consider regular a-gon in y,z-plane containing the point (0,1,0)



Idea: Rotation by 180° around the y-axis acts as reflection in y,z-plane

$$\begin{pmatrix} \cos 180° & & -\sin 180° \\ & 1 & \\ \sin 180° & & \cos 180° \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

TRICK.

Then we have

$$D_{2a} = \left\langle \left(\begin{array}{c|cc} 1 & & \\ \hline & \cos(\frac{2\pi}{a}) & -\sin(\frac{2\pi}{a}) \\ & \sin(\frac{2\pi}{a}) & \cos(\frac{2\pi}{a}) \end{array}\right), \left(\begin{array}{c|cc} -1 & & \\ \hline & 1 & \\ & & -1 \end{array}\right) \right\rangle$$

Three "Exceptional" cases:

(3) T = 12 rotations of a tetrahedron

(4) O = 24 rotations of cube/octahedron

(5) I = 60 rotations of icos./dodecahedron.

Theorem: That's All.

How can we prove it?

Let $G < SO(3)$ be finite and consider the set of axes

$$\{ l \subseteq \mathbb{R}^3 : l = \ker(A - I) \text{ for some } \underset{A \neq I}{A \in G} \}$$

Each axis intersects the sphere S^2 in two "poles". Let

$\mathcal{P}$ = the set of poles

Note that G acts on $\mathcal{P}$.

If $g(p) = p$ for some $p \in P$, $1 \neq g \in G$.

Then for any $h \in SO(3)$, let $q = h(p)$.
Since

$$hgh^{-1}(q) = hg(p) = h(p) = q$$

and $1 \neq hgh^{-1} \in G$, we have $q \in P$ ///

Thus P decomposes into G-orbits

$$P = \mathrm{Orb}_1 \cup \mathrm{Orb}_2 \cup \cdots \cup \mathrm{Orb}_m$$

Let $r_i := |\mathrm{stab}_G(p)|$ for $p \in \mathrm{Orb}_i$.

By counting the set

$$S = \{(g, p) : 1 \neq g \in G, \ p \in P, \ g(p) = p\}$$

in <u>two ways</u>, we get

$$(*) \quad 2(|G| - 1) = \sum_{p \in P} (r_p - 1)$$

choose g first — choose p first

Now let $o_i = |Orb_i|$, so

$$\sum_{p \in P} (r_p - 1) = \sum_{i=1}^{m} o_i (r_i - 1)$$

$$= \sum (o_i r_i - o_i)$$

$$= \sum \left(|G| - \frac{|G|}{r_i} \right)$$

$$= |G| \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right)$$

Then from (*) we have

$$2(|G| - 1) = |G| \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right)$$

(**) $$\boxed{2 - \frac{2}{|G|} = \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right)}$$

What is m? # orbits

By definition note that

$$r_i \geq 2$$

$$\Rightarrow 1 - \frac{1}{r_i} \geq \frac{1}{2} \quad \forall\, i = 1, \ldots, m$$

Then $m = 1$ is impossible since

$$1 \le 2 - \frac{2}{|G|} = 1 - \frac{1}{r_1} < 1 \quad \times$$

And $m \geq 4$ is impossible since

$$2 > 2 - \frac{2}{|G|} = \sum_{i=1}^{m} \left(1 - \frac{1}{r_i}\right) \geq 4 \cdot \frac{1}{2} = 2 \quad \times$$

So there are two cases

Case $m = 2$ (Two orbits):

$$\cancel{2} - \frac{2}{|G|} = \left(\cancel{1} - \frac{1}{r_1}\right) + \left(\cancel{1} - \frac{1}{r_2}\right)$$

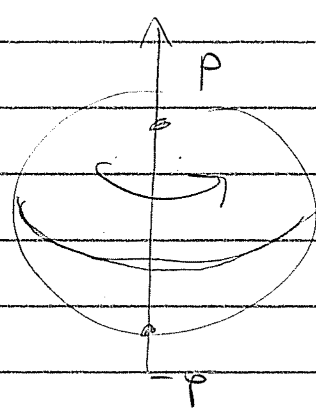
$$\frac{1}{r_1} + \frac{1}{r_2} = 2/|G|.$$

Since $r_1, r_2 \le |G|$, must have

$$r_1 = r_2 = |G| \Longrightarrow o_1 = o_2 = 1.$$

$Orb_1 = \{p\}$

$Orb_2 = \{-p\}$



$\Longrightarrow$ G is cyclic.

Case $m = 3$ (Three orbits).

$$2 - \frac{2}{|G|} = \left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right) + \left(1 - \frac{1}{r_3}\right)$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 + \frac{1}{|G|} > 1.$$

Solutions (r_1, r_2, r_3).

$(2, 2, a \geq 2) \Longrightarrow G \approx D_{2a}$.

(Exercise).

$(2, 3, 3) \Longrightarrow G \approx T$

$(2, 3, 4) \Longrightarrow G \approx O$

$(2, 3, 5) \Longrightarrow G \approx I$

$\blacksquare$