Thurs Apr 18

Weyl Groups

Last time we saw that unbranched Coxeter diagrams classify regular polytopes.

Today we consider a different class of diagrams.

Recall that a group $G < O(n)$ is called crystallographic if it preserves a full-rank lattice $\mathbb{Z}^n \approx \Lambda < \mathbb{R}^n$.

Recall that a "rotation" $\in O(n)$ is a product of two reflections.

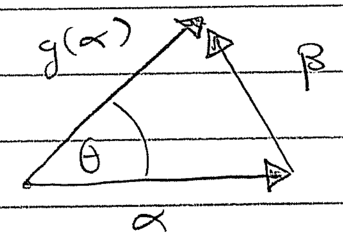
Theorem (Crystallographic Restriction):

Let $G < O(n)$ be crystallographic, preserving the lattice $\Lambda < \mathbb{R}^n$. Then every rotation in G has order

$$1, 2, 3, 4 \text{ or } 6.$$

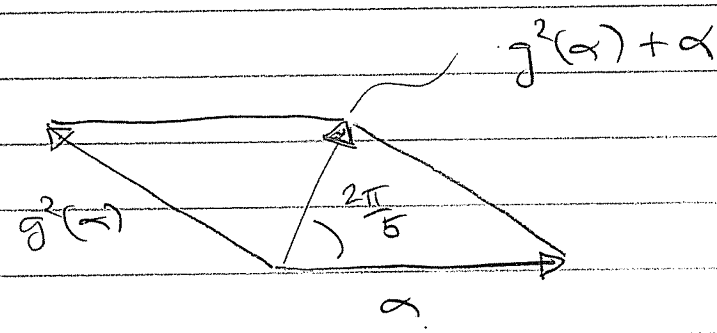
Proof: Let $g \in G$ be a rotation with angle θ. Let $\alpha \in \Lambda$ be a nonzero vector of minimal length

Consider the picture



Since $g(\alpha) \in \Lambda$ we have $\beta = g(\alpha) - \alpha \in \Lambda$.
By minimality of α we have $|\beta| \geq |\alpha|$.
Hence $\theta \geq 2\pi/6$. This implies that
$\langle g \rangle < O(2)$ is discrete, hence finite,
hence $\theta = 2\pi/m \geq 2\pi/6$. Then g
has order $m \leq 6$.

But $m = 5$ is impossible since in that
case $g^2(\alpha) + \alpha \in \Lambda$ is shorter than α:



Explicitly, we have

$$|g^2(\alpha) + \alpha| \approx (0.79)|\alpha|$$

$\blacksquare$

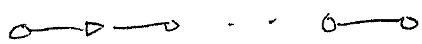
[Fun Fact: The numbers $m = 1, 2, 3, 4, 6$ are the solutions to

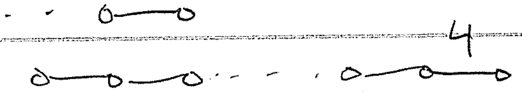
$$\varphi(m) \leq 2. \text{ (Euler's totient)}.$$]

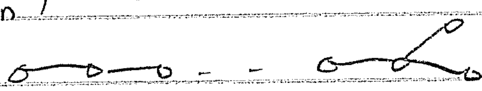
Definition: A crystallographic FGGR is called a Weyl group.

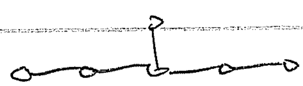
Theorem (Cartan-Killing, 1890):

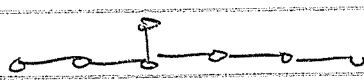
The Weyl groups are exactly

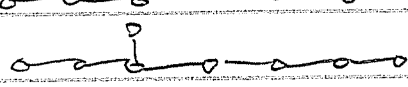
$A_n =$ 

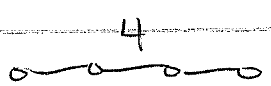
$B_n (= C_n) =$ 

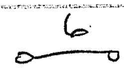
$D_n =$ 

$E_6 =$ 

$E_7 =$ 

$E_8 =$ 

$F_4 =$ 

$G_2 =$ 

And That's All.

Proof: (1) Let α_i, α_j be simple roots. The edge label between them is m_{ij}, where π/m_{ij} is the angle between the hyperplanes $H_{\alpha_i}, H_{\alpha_j}$. Then the order of the rotation $t_{\alpha_i} t_{\alpha_j}$ is m_{ij} and we conclude that $m_{ij} \in \{2, 3, 4, 6\}$.

Thus G has Coxeter diagram in the given list.

(2) Conversely, for each given Coxeter diagram we must construct a lattice, preserved by G.

We'll do this now.

Let Φ be the root system of G, with unit vectors. We want to change the lengths of the roots so that

$$\frac{2(\alpha \circ \beta)}{(\alpha \circ \alpha)} \in \mathbb{Z} \quad \forall \alpha, \beta \in \Phi,$$

while preserving the root system axioms:

- $\mathbb{R}\alpha \cap \Phi = \{\pm \alpha\} \quad \forall \alpha \in \Phi$
- $t_\alpha(\beta) \in \Phi \quad \forall \alpha, \beta \in \Phi$

Claim: It's enough to show this for the simple roots.

Proof sketch: Define $\langle \alpha, \beta \rangle := 2(\alpha \cdot \beta)/(\alpha \cdot \alpha)$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and assume that $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z} \ \forall i, j$.

(i) Show that

$$\langle \alpha, \beta \rangle = \langle g(\alpha), g(\beta) \rangle \ \forall g \in G, \ \alpha, \beta \in \Phi$$

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(ii) Note that $\forall \rho \in \Phi$, $\exists g \in G$ such that $g(\rho)$ is simple.

Since g^{-1} is a product of simple reflections, say $g^{-1} = s_1 s_2 \cdots s_k$, it follows by induction that

$$\rho = s_1 s_2 \cdots s_k (g(\rho))$$

is a $\mathbb{Z}$-linear combination of simple roots.

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(iii) Choose any $\rho, \mu \in \Phi$ and choose $g \in G$ such that $g(\rho) =: \alpha$ is simple. Then

$$\langle \rho, \mu \rangle = \langle g(\rho), g(\mu) \rangle$$
$$= \langle \alpha, g(\mu) \rangle$$
$$= \langle \alpha, \sum_i c_i \alpha_i \rangle \text{ with } c_i \in \mathbb{Z}.$$

$$= \frac{2(\alpha \cdot \sum_i c_i \alpha_i)}{(\alpha \cdot \alpha)} = \sum_i c_i \frac{2(\alpha \cdot \alpha_i)}{(\alpha, \alpha)}$$
$$= \sum_i c_i \langle \alpha, \alpha_i \rangle \in \mathbb{Z}$$

The claim is proved. $\boxtimes$

So our goal is:

Choose the lengths of the simple roots $\alpha_1, \ldots, \alpha_n$ so that

$$\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z} \quad \forall\, i, j.$$

So define new simple roots $\alpha_i' := c_i \alpha_i$ for real scalars $c_i \in \mathbb{R}$.

Recall that $m_{ij} \in \{1, 2, 3, 4, 6\}$.
We have

$$m_{ii} = 1 \implies \langle \alpha_i, \alpha_i \rangle = \frac{2(\alpha_i \cdot \alpha_i)}{(\alpha_i \cdot \alpha_i)} = 2 \in \mathbb{Z}.$$

$$m_{ij} = 2 \implies \langle \alpha_i, \alpha_j \rangle = \frac{2\cancelto{0}{(\alpha_i \cdot \alpha_j)}}{\cancel{(\alpha_i \cdot \alpha_i)}} = 0 \in \mathbb{Z}$$

So no problem. We are done if we can choose c_i such that

$$m_{ij} = 3 \implies c_i = c_j$$

$$m_{ij} = 4 \implies c_i = \sqrt{2}\, c_j \ \underline{\text{or}}\ c_j = \sqrt{2}\, c_i$$

$$m_{ij} = 6 \implies c_i = \sqrt{3}\, c_j \ \underline{\text{or}}\ c_j = \sqrt{3}\, c_i.$$

And this is always possible because the Coxeter diagram of G has no cycles

Now we complete the proof of Cartan–Killing's result.

Given G with $m_{ij} \in \{1, 2, 3, 4, 6\}$, choose a root system Φ for G such that

- $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ $\quad \forall\, \alpha \in \Phi$
- $t_\alpha(\beta) \in \Phi$ $\quad \forall\, \alpha, \beta \in \Phi$
- $\langle \alpha, \beta \rangle \in \mathbb{Z}$ $\quad \forall\, \alpha, \beta \in \Phi$.

This is called a crystallographic root system.

Now define the root lattice

$$\Lambda := \left\{ \sum_i c_i \alpha_i \; : \; c_i \in \mathbb{Z}, \alpha_i \in \Phi \right\}$$

This is a full-rank lattice $\cong \mathbb{Z}^n$ with basis of simple roots.

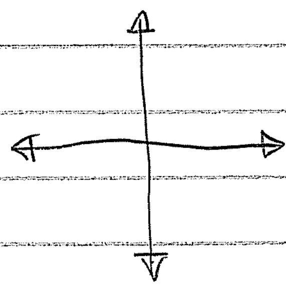
Finally, G preserves Λ: Given $\lambda = \sum_i c_i \alpha_i \in \Lambda$ and $g = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k} \in G$ we have

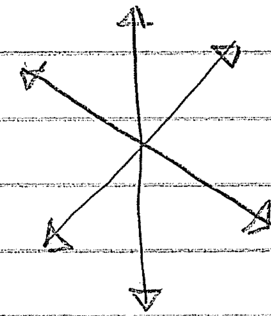
$$\begin{aligned} g(\lambda) &= t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{k-1}} (t_{\alpha_k}(\lambda)) \\ &= t_{\alpha_1} \cdots t_{\alpha_{k-1}} (\lambda - \langle \alpha_k, \lambda \rangle \alpha_k) \\ &= t_{\alpha_1} \cdots t_{\alpha_{k-1}} (\lambda) - \langle \alpha_k, \lambda \rangle t_{\alpha_1} \cdots t_{\alpha_{k-1}} (\alpha_k) \end{aligned}$$

$\in \Lambda$ by induction.

QED.

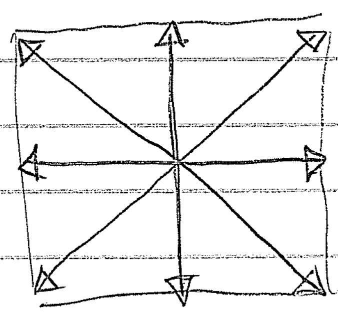
Example: The crystallographic root systems of rank 2 are exactly



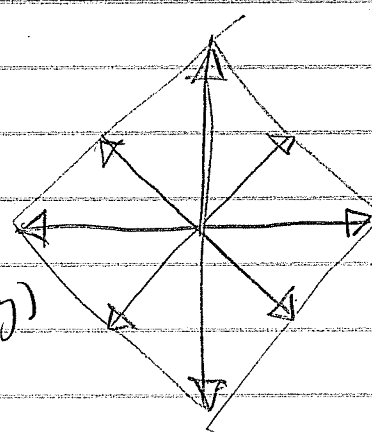


$A_1 \times A_1 = \circ\overset{2}{\longrightarrow}\circ$
$= \circ \quad \circ$

$A_2 = \circ\overset{3}{\longrightarrow}\circ = \circ\!\longrightarrow\!\circ$

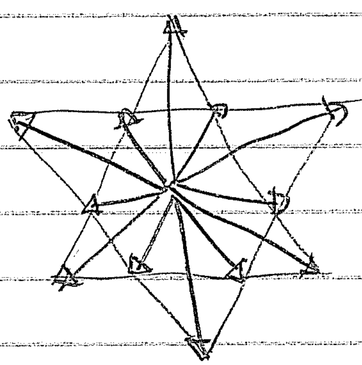


$\approx$
(accidentally)



$B_2 = \circ\overset{4}{\rightarrow}\circ$

$C_2 = \circ\overset{4}{\leftarrow}\circ$

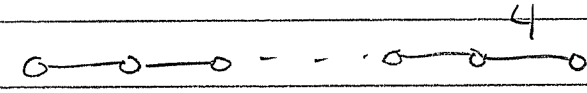


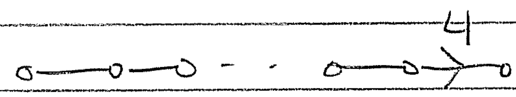
$G_2 = \circ\overset{6}{\longrightarrow}\circ$

And That's All!

[Remark: Only regular triangles, squares and hexagons can tile the plane!]

Warning: The crystal root system of a Weyl group is not unique.

The FGGR 
has two nonisomorphic root systems, called

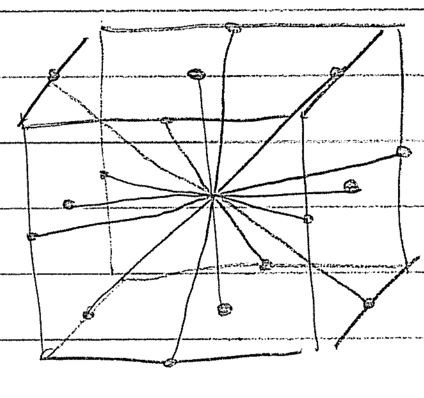
B_n = 

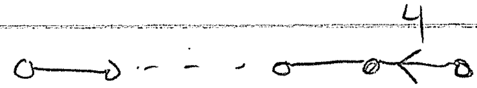
$$\Phi = \{\pm e_i : 1 \le i \le n\} \cup \{\pm e_i \pm e_j : 1 \le i < j \le n\}$$

$$\Phi^+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i < j\} \cup \{+e_i\}.$$

$$\Pi = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}.$$

Picture: Hypercube.



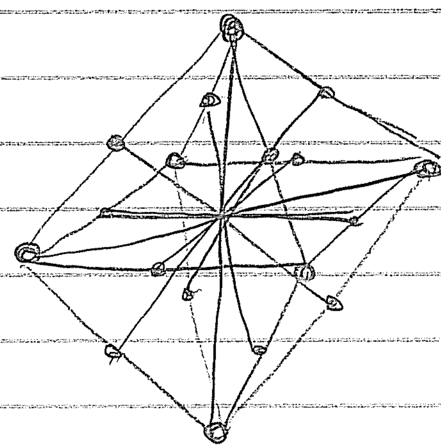
$C_n =$ 

$\Phi = \{\pm 2e_i\} \cup \{\pm e_i \pm e_j : i<j\}$.

$\Phi^+ = \{e_i - e_j : i<j\} \cup \{e_i + e_j : i<j\} \cup \{+2e_i\}$.

$\Pi = \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, 2e_n\}$

Picture: Hyperoctahedron



In general, given crystal. Φ, define the <u>coroot</u> <u>system</u>

$$\Phi^\vee := \left\{ \alpha^\vee := \left(\frac{2}{\alpha \circ \alpha}\right)\alpha \;:\; \alpha \in \Phi \right\}$$

<u>Note</u>: $B_n^\vee = C_n$
$C_n^\vee = B_n$.

Why do we care?

Theorem (Carton-Killing-Weyl- ...):

There is a correspondence between families of nice Lie groups and crystallographic root systems.

A_{n+1} $GL(n)$, $SL(n)$, $SU(n)$.

B_n $SO(2n+1)$

C_n $Sp(n)$

D_n $SO(2n)$.

+ some exceptional Lie groups, called

$$\left.\begin{matrix} F_4 \\ E_6 \\ E_7 \\ E_8 \end{matrix}\right\} \longleftarrow \text{Octonians (Vinberg)}$$

G_2 = symmetries of the octonians