Thurs Jan 17

First Goal:
Classify discrete subgroups of $SO(3)$.
Spoiler: They are in bijection with triples $p, q, r \in \mathbb{N}$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

(Look familiar?)

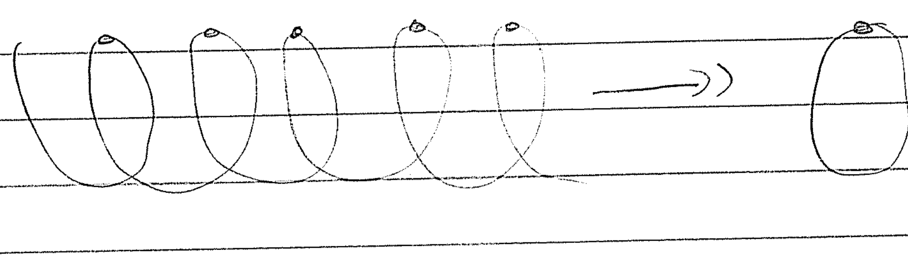
Start small:

$O(1) = \{\pm 1\}$
$SO(1) = \{+1\}$

$$U(1) = \{z \in \mathbb{C} : |z| = 1\} \sim S^1 \quad \text{circle.}$$

Recall the covering map

$$1 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow U(1) \longrightarrow 1$$



We used this to show that the discrete subgroups of $U(1)$ are the groups

$$\mathcal{U}_a := \{ e^{2\pi i k/a} : k = 1, 2, \ldots, a \}$$

"ath roots of unity"

for $a \in \mathbb{N}$

[Remark: Since $U(1)$ is compact, its discrete subgroups are finite.]

Next up: $O(2)$.

$$O(2) = \{ A \in Mat_2(\mathbb{R}) : A^t A = I \}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$A^t A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} (a\ c)\begin{pmatrix} a \\ c \end{pmatrix} & (a\ c)\begin{pmatrix} b \\ d \end{pmatrix} \\ (b\ d)\begin{pmatrix} a \\ c \end{pmatrix} & (b\ d)\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

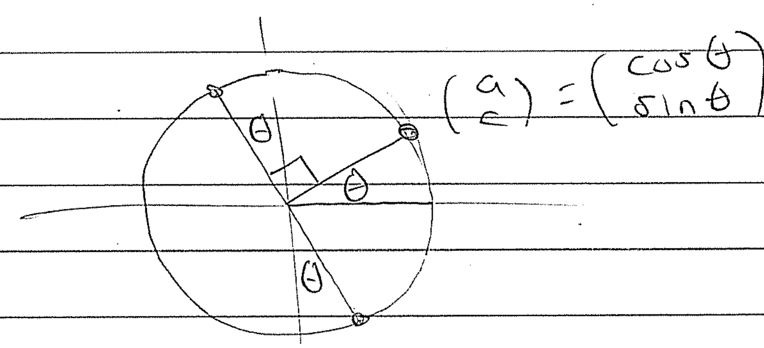
$\Longrightarrow$ $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$

are orthogonal unit vectors.

Without loss, say

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ for some } \theta \in \mathbb{R}.$$

Picture:



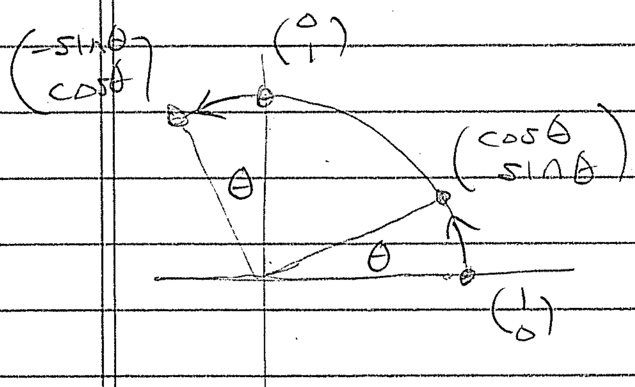
Then $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ OR $\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

only 2 choices!

Let $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\det A_\theta = \cos^2\theta + \sin^2\theta = +1.$

What does it do?

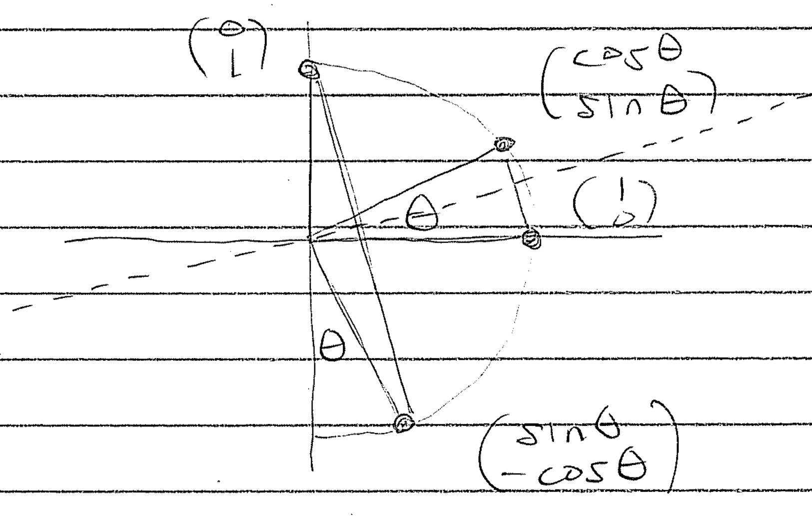


Rotation by θ c.c.w.

Let $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

$$\det R_\theta = -\cos^2\theta - \sin^2\theta = -1$$

What does it do?



Reflect across line of angle $\frac{\theta}{2}$.

So we have

$$SO(2) = \{ A_\theta : \theta \in \mathbb{R} \}$$

$$O(2) = \{ A_\theta, R_\theta : \theta \in \mathbb{R} \}$$

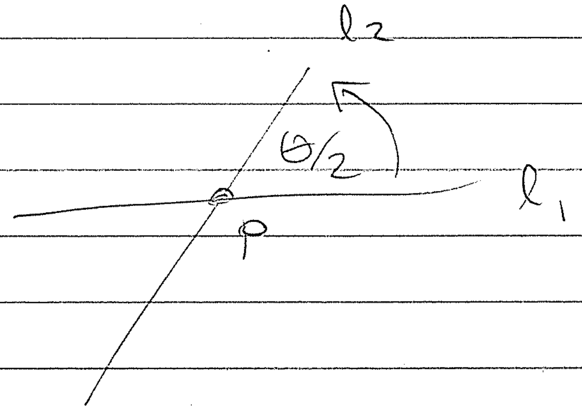
Angle sum/difference identities
$\Longleftrightarrow$ group relations

① $A_\alpha A_\beta = A_{\alpha+\beta}$

Note: $A_\alpha^{-1} = A_{-\alpha}$

② $R_\alpha R_\beta = A_{\alpha-\beta}$

[In general, consider two lines



Then $Ref_{l_2} \circ Ref_{l_1} = Rot_\theta^p$.]

Note: $R_\alpha^{-1} = R_\alpha$.

(3) $R_\alpha A_\beta = R_{\alpha-\beta}$

(4) $A_\alpha R_\beta = R_{\alpha+\beta}$

In particular, note

$$R_\alpha A_\beta R_\alpha = A_{-\beta} \quad \forall \alpha, \beta \in \mathbb{R}$$

So we have $SO(2) \triangleleft O(2)$

In fact

$$O(2) = SO(2) \rtimes \langle R \rangle \cong SO(2) \rtimes \mathbb{Z}/2\mathbb{Z}$$

for any reflection R.

Theorem (Euler):
We have an isomorphism

$$U(1) \cong SO(2)$$
$$e^{i\theta} \longleftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

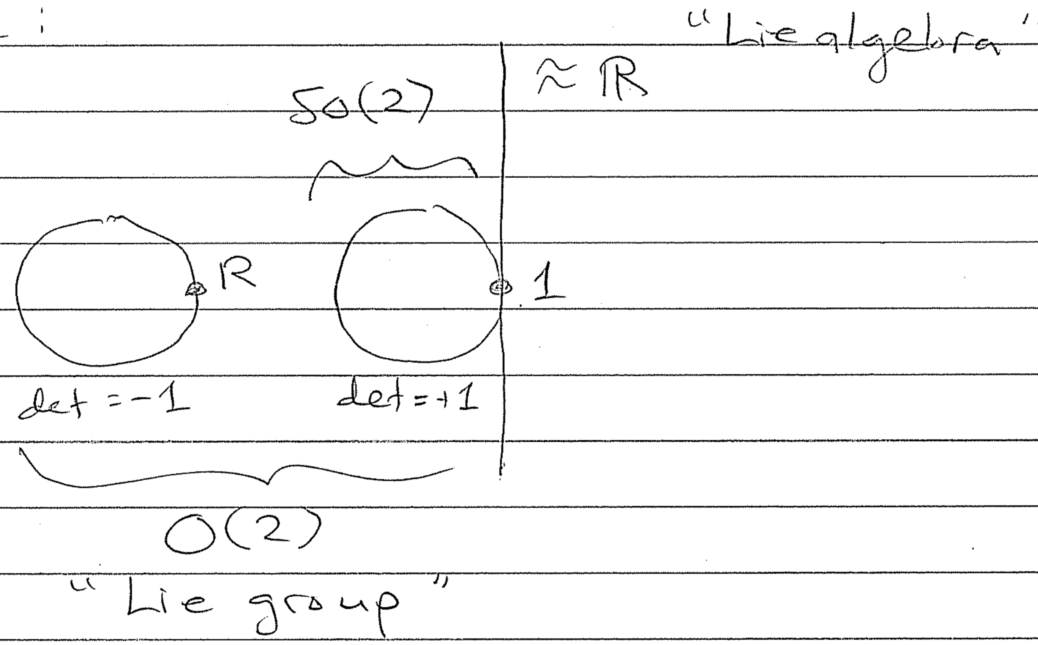
"anomolous isogeny"

Topologically we have

$$O(2) \sim S^1 \cup S^1$$

because $\det : O(2) \to \mathbb{R}$ is continuous

Picture:



[Slogan: Lie algebra can't see topology.]

Q: Discrete (i.e. finite) subgroups of $O(2)$?

Let $H < O(2)$ be discrete.

If $H < SO(2) \cong U(1)$ then we know

$$H \cong \mathbb{Z}/a\mathbb{Z}.$$

Otherwise, H contains a reflection R.

Then let $H' := H \cap SO(2) \cong \mathbb{Z}/a\mathbb{Z}$

for all $A_\theta \in H'$ we have

$$R A_\theta R = A_{-\theta} = A_\theta^{-1} \in H'.$$

Hence $H' \triangleleft H$ and we conclude

$$H = H' \rtimes \langle R \rangle$$

$$\cong \mathbb{Z}/a\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = D_{2a}.$$

"Dihedral group"

Summary: Discrete subgroups of $O(2)$ are cyclic or dihedral

Explicitly, for each $a \in \mathbb{N}$ we have

$$D_{2a} = \left\langle \underset{R_{x\text{-axis}}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \underset{A_{2\pi/a}}{\begin{pmatrix} \cos(2\pi/a) & -\sin(2\pi/a) \\ \sin(2\pi/a) & \cos(2\pi/a) \end{pmatrix}} \right\rangle$$

So far so good. What's next?

$U(2)$, $SU(2)$, $O(3)$, $SO(3)$.

Next: $SO(3)$.
What can we say?

Cartan-Dieudonné says every $A \in O(3)$ is a product of ≤ 3 reflections in planes through O.

	# reflections	det	geometry.
	0	+1	id
$SO(3)$	1	−1	reflection
	2	+1	rotation around an axis
	3	−1	screw reflection

That's all.

This Implies

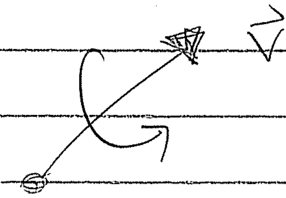
Theorem (Euler, 1776):

If A and B are rotations in $\mathbb{R}^3$, fixing O, then so is $A \circ B$.
[NOT intuitively obvious!]

He didn't use Cartan-Dieudonné.

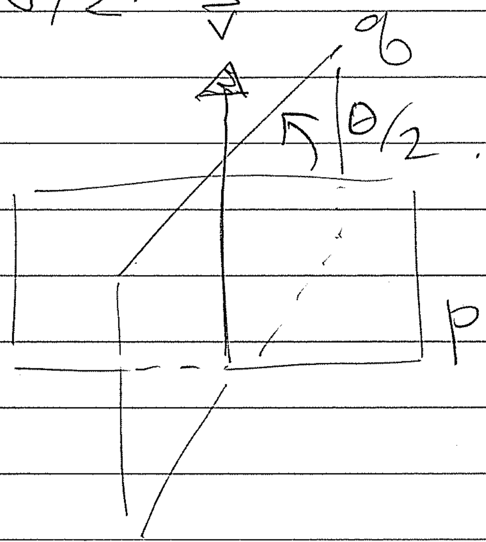
Elementary Proof:

Given vector $\vec{v} \in \mathbb{R}^3$ let $A_\theta^{\vec{v}}$ be rotation around $\vec{v}$ by θ c.c.w.



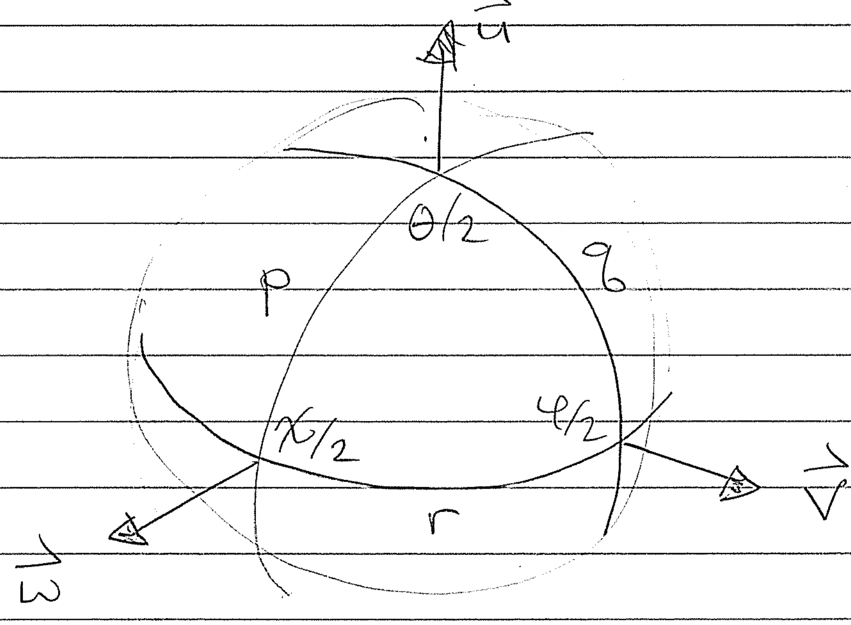
Given a plane $p \subseteq \mathbb{R}^3$ let R_p be reflection across the plane.

If planes p, q intersect in $\vec{v}$ with angle $\theta/2$:



Note that $R_q \circ R_p = A_\theta^{\vec{v}}$
(as before)

Finally, consider any spherical triangle:



Then we have

$$A^{\vec{v}}_{\varphi} \circ A^{\vec{u}}_{\theta} = (R_r \circ R_q) \circ (R_q \circ R_p)$$

$$= R_r \circ \left(\cancel{R_q \circ R_q}\right) \circ R_p$$

$$= R_r \circ R_p$$

$$= A^{\vec{w}}_{-\chi}$$

∎

$SO(3)$ = group of rotations of the sphere.

Q: Discrete subgroups?