Thurs Mar 21

Today: My Favorite Matrix

Consider the $\mathbb{R}$-vector space $C^\infty[0, n+1]$ of C^∞ functions $f: [0, n+1] \to \mathbb{R}$.
The derivative is a linear function

$$D: C^\infty[0,n+1] \longrightarrow C^\infty[0,n+1]$$

because $(f+\alpha g)' = f' + \alpha g'$, as you know.

Problem: Find the eigenfunctions of the 2nd derivative $D^2 = \Delta$ (AKA the Laplacian), subject to boundary conditions

$$f(0) = f(n+1) = 0.$$

The general solution of $\Delta f = \lambda f$ is

$$f(x) = A \sin(x\sqrt{-\lambda}) + B \cos(x\sqrt{-\lambda}).$$

Hence λ must be negative.
Say $\lambda = -L^2$

$$f(x) = A \sin(Lx) + B \cos(Lx).$$

Boundary Conditions:

$$f(0) = 0 \implies A\sin(0) + B\cos(0) = 0$$
$$\implies B = 0.$$

$$f(n+1) = 0 \implies A\sin(L(n+1)) = 0$$

$$\implies L(n+1) = \pi k \text{ for some } k \in \mathbb{Z}.$$

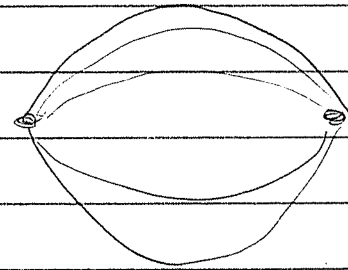
Hence the linearly independent eigenfunctions are, for all $k \in \mathbb{N}$:

$$\lambda_k = \frac{-\pi^2 k^2}{(n+1)^2} \quad \& \quad f_k(x) = A\sin\left(\frac{\pi k x}{n+1}\right)$$

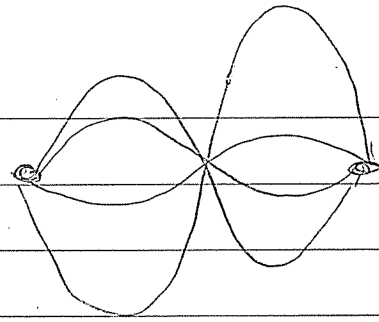
Picture:

$k = 0$

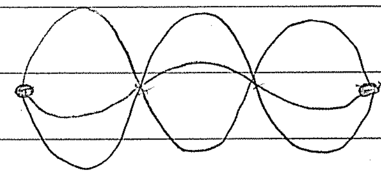
$k = 1$



$k=2$



$k=3$



Now put in time $f(x) = \varphi(x, 0)$ for some $\varphi(x,t)$. The <u>wave</u> <u>equation</u> says

$$\boxed{\frac{d^2 \varphi}{dt^2} = c^2 \frac{d^2}{dx^2} \varphi} \qquad (*)$$

where $c = \sqrt{\dfrac{\text{tension}}{\text{density}}}$ = speed of the wave

Boundary Conditions

$$\varphi(0,t) = \varphi(n+1, t) = 0 \quad \forall \text{ time } t$$

To solve, we assume separ<u>ab</u>ility

$$\varphi(x,t) = f(x)\, g(t).$$

$\left[\text{Note } f(0) = f(n+1) = 0\right]$.

Then (*) becomes

$$f(x)g''(t) = c^2 f''(x) g(t).$$

$$\Longrightarrow \frac{1}{c^2}\frac{g''(t)}{g(t)} = \frac{f''(x)}{f(x)} = \lambda$$

must be constant.

$$\Longrightarrow \begin{cases} f''(x) = \lambda f(x) & (1) \\ g''(t) = c^2\lambda\, g(t) & (2) \end{cases}$$

We already know that

$$\lambda = -\frac{\pi^2 k^2}{(n+1)^2} \quad \& \quad f(x) = A \sin\left(\frac{\pi k x}{n+1}\right)$$

Then $c^2\lambda = -\dfrac{c^2\pi^2 k^2}{(n+1)^2}$, so

$$g(t) = C \sin\left(\frac{\pi k t}{c(n+1)}\right) + D\cos\left(\frac{\pi k t}{c(n+1)}\right)$$

Oscillation amplitude $C^2 + D^2$

frequency $k/(2c(n+1))$

phase $\tan^{-1}\left(\frac{D}{C}\right)$.

Lucky: The separable solutions form a complete basis of solutions.

Hence the general solution is

$$\sum_{k=0}^{\infty} \left[A_k \sin\left(\frac{\pi k t}{c(n+1)}\right) + B_k \cos\left(\frac{\pi k t}{c(n+1)}\right) \right] \sin\left(\frac{\pi k x}{n+1}\right)$$

"Fourier Series"

where $A_k^2 + B_k^2$ is the energy in the "k-th harmonic" and $\sum_k (A_k + B_k)^2 < \infty$

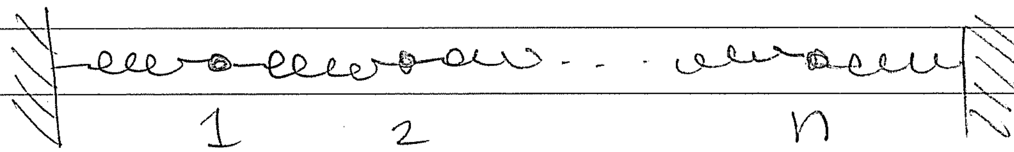
Q: Where does the wave equation come from?

Here's a derivation due to Euler/Lagrange based on

"Researches on the Nature and Propagation of Sound", 1759

by Lagrange.

Consider n point masses and $n+1$ springs



Let x_i = rightward displacement of mass i

e_i = elongation of spring i

$$\vec{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\vec{x}.$$

Let y_i = spring force from spring i.

$$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_{n+1} \end{pmatrix}}_{\text{spring constants.}} \begin{pmatrix} e_1 \\ \vdots \\ e_{n+1} \end{pmatrix} = C\vec{e}$$

Assume $C = -I$

Let F_i = spring force felt by i^{th} mass.

$$\vec{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} = A^T \vec{y}$$

Putting it together:

$$\begin{aligned} \vec{F} &= A^T \vec{y} \\ &= A^T C \vec{e} \\ &= A^T C A \vec{x} \\ &= A^T (-I) A \vec{x} \\ &= -A^T A \vec{x} \end{aligned}$$

$$= -\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \vec{x}$$

Finally, Newton says

$$\vec{F} = \underbrace{\begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}}_{\text{Assume } = I} \frac{d^2}{dt^2} \vec{x}$$

$$\vec{F} = \frac{d^2}{dt^2}\vec{x}$$

$$\boxed{-A^T A\,\vec{x} = \frac{d^2}{dt^2}\vec{x}}$$

Discrete Wave Equation.

Q: $-A^T A \approx \dfrac{d^2}{dx^2}$?

("Discrete Laplacian")

For continuous $f: \mathbb{R} \to \mathbb{R}$,

$$f'(x) \approx \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}$$

for small h.

$f''(x) = g'(x)$ where $g'(x)$

$$\approx \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}.$$

$$\approx \frac{\dfrac{f(x+\frac{h}{2}+\frac{h}{2}) - f(x+\frac{h}{2}-\frac{h}{2})}{h} - \dfrac{f(x-\frac{h}{2}+\frac{h}{2}) - f(x-\frac{h}{2}-\frac{h}{2})}{h}}{h}$$

$$= \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h) \right].$$

Let's put $h = 1$ and write a function as a vector

$$f = \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{pmatrix}.$$

Assume $f(0) = f(n+1) = 0$. Then

$$f''(x) \approx \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} f = -A^T A f$$

In the limit $h \to 0$ we have

$$-A^T A f \longrightarrow f''(x).$$

Can we solve $-A^T A \vec{x} = \frac{d^2}{dt^2} \vec{x}$?

Suppose $-A^T A$ has evectors $\vec{x}_1, \dots, \vec{x}_n$ with evalues $\lambda_1, \dots, \lambda_k < 0$.
Let $\omega_k = \sqrt{-\lambda_k}$. Then the general solution is

$$\sum_{k=1}^{n} \left[A_k \sin(\omega_k t) + B_k \cos(\omega_k t)\right] \vec{x}_k.$$

"Finite Fourier Transform"

To compute the $\vec{x}_k, \lambda_k$:

Note that

$$A^T A = 2\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}$$

$$= 2I - \text{Path}.$$

Suppose $\text{Path}\,\vec{x}_k = \mu_k \vec{x}_k$.

$\downarrow$

Then $A^T A \vec{x}_k = (2I - \text{Path})\vec{x}_k$

$$= 2\vec{x}_k - \mu_k \vec{x}_k$$
$$= (2-\mu_k)\vec{x}_k.$$

Finally, recall the trig identity

(*) $$\sin\left(\frac{(l-1)k\pi}{n+1}\right) + \sin\left(\frac{(l+1)k\pi}{n+1}\right) = \left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\sin\left(\frac{lk\pi}{n+1}\right)$$

for all $k, l \in \mathbb{Z}$.

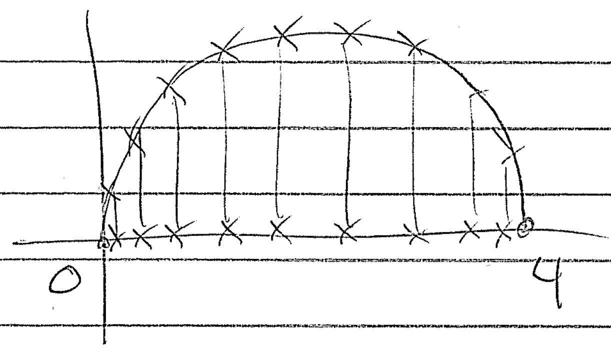
If we define $\vec{x}_k = \begin{pmatrix} \sin\left(\frac{k\pi}{n+1}\right) \\ \sin\left(\frac{2k\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nk\pi}{n+1}\right) \end{pmatrix}$

Then (*) says $\text{Path}\,\vec{x}_k = \underbrace{2\cos\left(\frac{k\pi}{n+1}\right)}_{\mu_k}\vec{x}_k.$

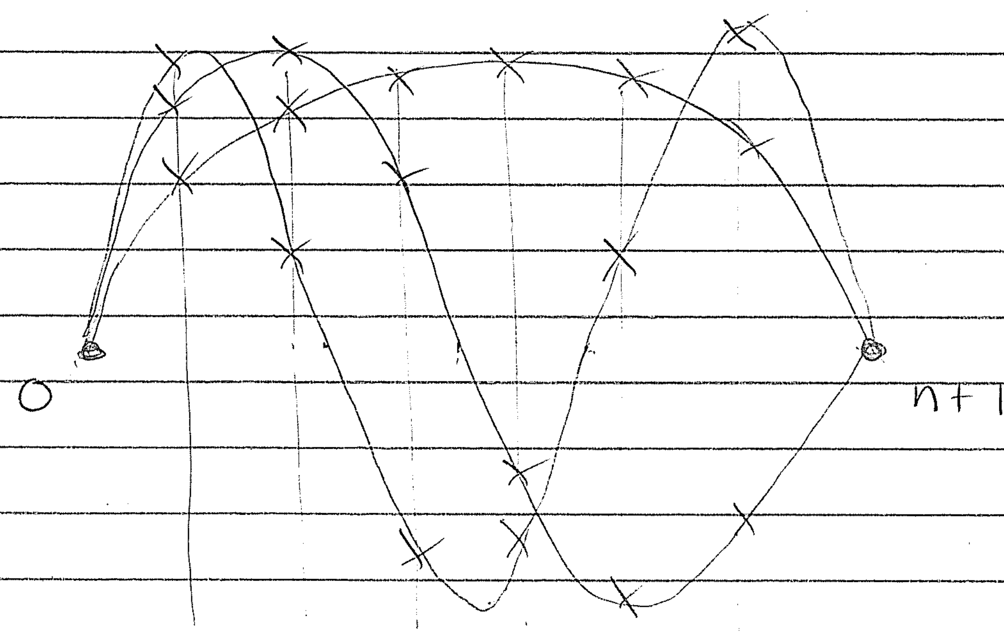
$$\Longrightarrow \lambda_k = 2 - \mu_k$$
$$= 2\left(1 - \cos\left(\frac{k\pi}{n+1}\right)\right) > 0$$

as desired.

Eigenvalues of A^TA:



Eigenvectors of A^TA:



The key property that made it work:

A^TA is positive definite ....

to be continued