Mon Mar 18

More About My Favorite Cone:

Consider the cone in $\mathbb{R}^n$ with basis

$$\alpha_1 = e_1 - e_2$$
$$\alpha_2 = e_2 - e_3$$
$$\vdots$$
$$\alpha_{n-1} = e_{n-1} - e_n.$$

It lives in the subspace

$$\mathbb{R}^n_0 := (e_1 + \cdots + e_n)^\perp = \mathbb{1}^\perp$$

so we might as well project onto $\mathbb{R}^n_0$.
The matrix of the projection is

$$P = I - \frac{\mathbb{1}\mathbb{1}^t}{\mathbb{1}^t\mathbb{1}}.$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} - \frac{1}{n}\begin{pmatrix} 1 & 1 & \\ 1 & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

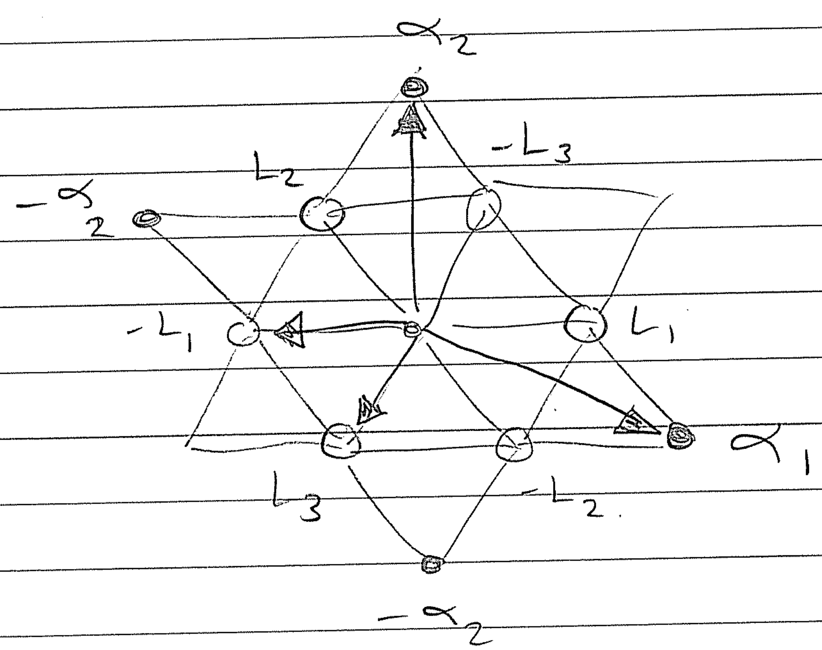
$$= \frac{1}{n}\begin{pmatrix} n-1 & & -1 \\ & n-1 & \\ & & \ddots & \\ -1 & & & n-1 \end{pmatrix}$$

The standard basis projects to

$$L_i := Pe_i = e_i - \frac{1}{n}\mathbb{1}$$

These $L_1, L_2, \ldots, L_n$ are the vertices of a regular simplex in $\mathbb{R}^n_0$, centered at O.

Picture: $\mathbb{R}^3_0$



It looks like the polar to cone(α_1, α_2) is cone$(-L_1, L_3)$ = cone$(L_2 + L_3, L_3)$.

and the dual cone is cone$(L_1, L_1 + L_2)$.

How to compute the dual cone in general?

Let $B : \mathbb{R}^n_0 \times \mathbb{R}^n_0 \to \mathbb{R}$ be the restriction of the dot product. What is the Gram matrix of B?

First we need a basis. Consider the "standard" basis

$$L_1, L_2, \ldots, L_{n-1} \in \mathbb{R}^n_0$$

Note that

$$B(L_i, L_j) = \left(e_i - \tfrac{1}{n}\mathbb{1}\right)\circ\left(e_j - \tfrac{1}{n}\mathbb{1}\right)$$
$$= e_i \circ e_j - \tfrac{1}{n} e_i \circ \mathbb{1} - \tfrac{1}{n}\mathbb{1}\circ e_j + \tfrac{1}{n^2}\mathbb{1}\circ\mathbb{1}$$

$$= \delta_{ij} - \frac{1}{n} - \frac{1}{n} + \frac{1}{n} = \delta_{ij} - \frac{1}{n}$$

So the Gram matrix is

$$[B]_{st} = [B(L_i, L_j)]_{i,j \in [n-1]} = I - \frac{1}{n} J$$

where

$$J = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}}_{n-1} \Bigg\} n-1$$

Consider the "root" basis

$$\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathbb{R}^n_0$$

and let $\omega_1, \omega_2, \ldots, \omega_{n-1} \in \mathbb{R}^n_0$ be the dual basis, defined by

$$B(\omega_i, \alpha_j) = \delta_{ij}$$

Then in "standard" coordinates we have

$$\begin{pmatrix} \omega_1^t \\ \vdots \\ \omega_{n-1}^t \end{pmatrix} [B]_{St} (\alpha_1 \cdots \alpha_{n-1}) = I.$$

Note.
$$\begin{aligned}
\alpha_1 &= L_1 - L_2 \\
\alpha_2 &= L_2 - L_3 \\
&\vdots \\
\alpha_{n-2} &= L_{n-2} - L_{n-1} \\
\alpha_{n-1} &= L_{n-1} - L_n \\
&= L_{n-1} - (-L_1 - L_2 - \cdots - L_{n-1}) \\
&= L_1 + L_2 + \cdots + L_{n-2} + 2L_{n-1}
\end{aligned}$$

$$\Longrightarrow (\alpha_1 \cdots \alpha_n) = \begin{pmatrix} 1 & & 0 & 1 \\ -1 & 1 & & \vdots \\ & -1 & \ddots & 1 & 1 \\ & & \ddots & & 1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

Hence $[B]_{St}(\alpha_1 \cdots \alpha_{n-1})$

$$= I(\alpha_1 \cdots \alpha_{n-1}) - \frac{1}{n} J(\alpha_1 \cdots \alpha_{n-1})$$

$$= I(\alpha_1 \cdots \alpha_{n-1}) - \frac{1}{n} \begin{pmatrix} 0 & \cdots & 0 & n \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & \vdots \\ & -1 & \ddots & & \vdots \\ & & \ddots & 1 & 1 \\ & & & -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & -1 & \ddots & 1 \\ 0 & & \ddots & -1 \quad 1 \end{pmatrix}$$

Finally we get

$$\begin{pmatrix} \omega_1^t \\ \vdots \\ \omega_{n-1}^t \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

That is: $\omega_1 = L_1$

$$\omega_2 = L_1 + L_2$$

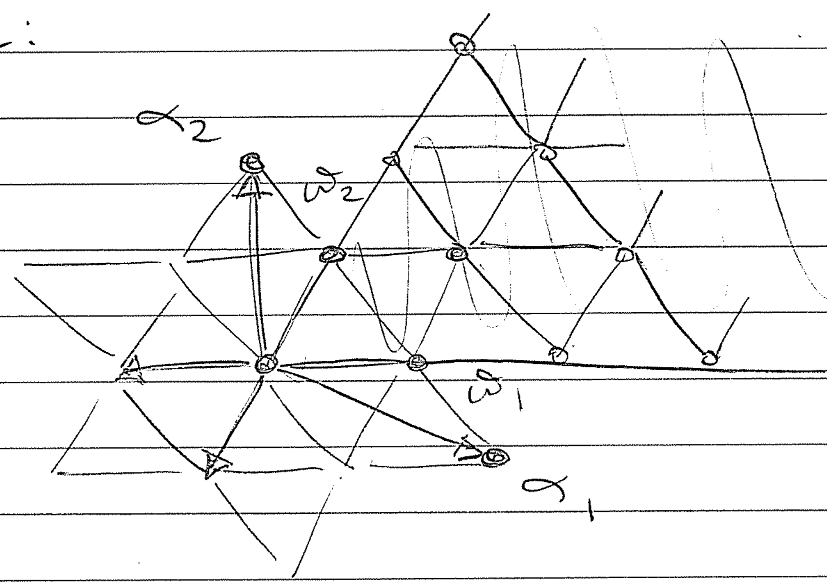
$$\vdots$$

$$\omega_{n-1} = L_1 + L_2 + \cdots + L_{n-1}$$

This is called the basis of "fundamental weights".

Elements of the integer cone $\mathbb{Z}^+\langle \omega_1, \ldots, \omega_{n-1} \rangle$ are called "dominant weights".

Picture:



Note that dominant weights biject to "Young diagrams" with at most $n-1$ rows.

Example: The dominant weight

$$2\omega_1 + 3\omega_2 + 1\omega_3$$
$$= 2L_1 + 3(L_1+L_2) + 1(L_1+L_2+L_3)$$

corresponds to "Young Diagram"

$$\begin{array}{|c|c|c|c|c|c|}
\hline
L_1 & L_1 & L_1 & L_1 & L_1 & L_1 \\
\hline
L_2 & L_2 & L_2 & L_2 & \multicolumn{2}{c}{} \\
\cline{1-4}
L_3 & \multicolumn{5}{c}{} \\
\cline{1-1}
\multicolumn{1}{c}{1} & \multicolumn{3}{c}{3} & \multicolumn{2}{c}{2}
\end{array}$$

☆ Important Theorem ☆

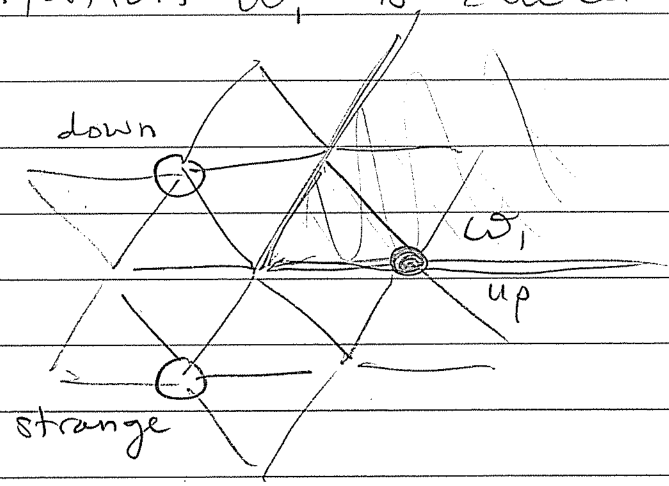
There is a bijection between dominant weights and irreducible representations of $SU(n)$. Furthermore, the dimension of the irrep corresponding to weight $a_1\omega_1 + a_2\omega_2 + \cdots + a_{n-1}\omega_{n-1}$ is

$$\prod_{1 \le i < j \le n} \frac{(a_i + \cdots + a_{j-1}) + j - i}{j - i}$$

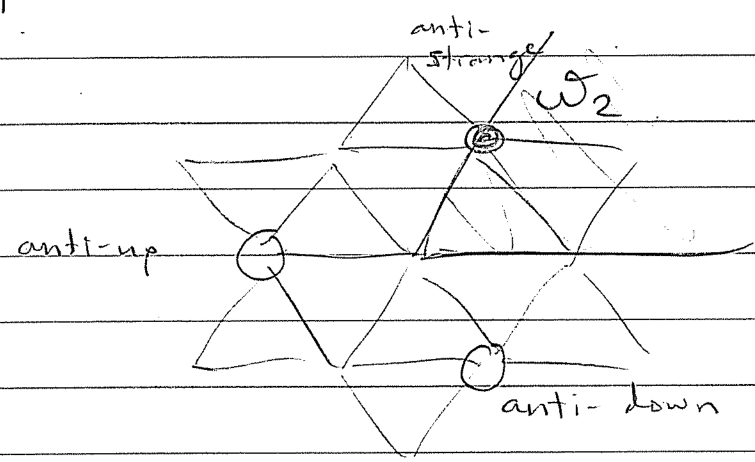
"Weyl Character Formula"

Example: Representations of $SU(3)$ are used to describe/define elementary particles.

Representation ω_1 is called "quarks"



Representation ω_2 is called "antiquarks"



These generate all the other representations / particles.

However, the "root" basis of $\mathbb{R}^n_0$ is more natural for geometry:

Let $R = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subseteq \mathbb{R}^n_0$

(Recall $\alpha_i = e_i - e_{i+1}$)

Let's compute ω_i in root coordinates.
The Gram matrix of B with respect to R is

$$[B]_R = [B(\alpha_i, \alpha_j)]_{i,j \in [n-1]} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

Then the relations $B(\omega_i, \alpha_j) = \delta_{ij}$ become

$$\begin{pmatrix} \omega_1^t \\ \vdots \\ \omega_{n-1}^t \end{pmatrix} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} (\alpha_1 \cdots \alpha_{n-1}) = I$$

where now we have

$$(\alpha_1 \cdots \alpha_{n-1}) = I$$

Hence $\begin{pmatrix} \omega_1^t \\ \vdots \\ \omega_{n-1}^t \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}^{-1}$

in root coords

Example: For $n=4$ we have

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Hence

$$\omega_1 = \frac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3)$$

$$\omega_2 = \frac{1}{4}(2\alpha_1 + 4\alpha_2 + 2\alpha_3)$$

$$\omega_3 = \frac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3).$$

See Zome model

Conversely, we have

$$\alpha_1 = 2\omega_1 - \omega_2$$

$$\alpha_2 = -\omega_1 + 2\omega_2 - \omega_3$$

$$\alpha_3 = -\omega_2 + 2\omega_3$$

See other Zome model.

Exercise: Show that

$$i,j \text{ entry of } \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}^{-1}$$

$$= \begin{cases} \frac{1}{n+1} j(n+1-i) & i \le j \\ \frac{1}{n+1} i(n+1-j) & j \le i \end{cases}$$