Thurs Mar 7

Recall: Given any subset $X \subseteq \mathbb{R}^n$, we define the dual cone

$$X^* = \{ u \in \mathbb{R}^n : u^t v \leq 0 \quad \forall\, v \in X \}$$

Certainly X^* is a convex cone because

- given $u_1, \dots, u_m \in X^*$ and $a_1, \dots, a_m \geq 0$ we have for all $v \in X$ that

$$(a_1 u_1 + \cdots + a_m u_m)^t v$$
$$= \underbrace{a_1 u_1^t v}_{\leq 0} + \underbrace{a_2 u_2^t v}_{\leq 0} + \cdots + \underbrace{a_m u_m^t v}_{\leq 0} \leq 0.$$

$$\Rightarrow a_1 u_1 + \cdots + a_m u_m \in X^* \qquad ///$$

Furthermore, if $C \subseteq \mathbb{R}^n$ is a finitely generated cone, say

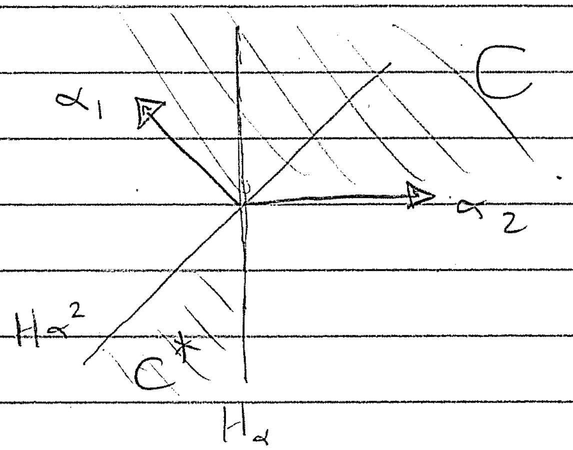
$$C = \mathbb{R}_+ \langle \alpha_1, \dots, \alpha_m \rangle$$

then C^* is the intersection of closed negative half-spaces

$$H^-_{\alpha_i} = \{ u \in \mathbb{R}^n : u^t \alpha_i \leq 0 \}.$$

Hence C^* is a polyhedral cone.

Picture:



In general, C^{**} is the convex closure of C, so if C is already convex closed we have

$$C^{**} = C.$$

The Duality Theorem for Cones:

Let C be a closed cone. Then

C is fin. gen. $\iff$ C^* is fin. gen.

Corollary (Farkas - Minkowski - Weyl): 1898 1896 1935

C is fin. gen $\iff$ C is polyhedral.

Proof: omitted (see handout) $\blacksquare$

But I will prove it for simplicial cones.

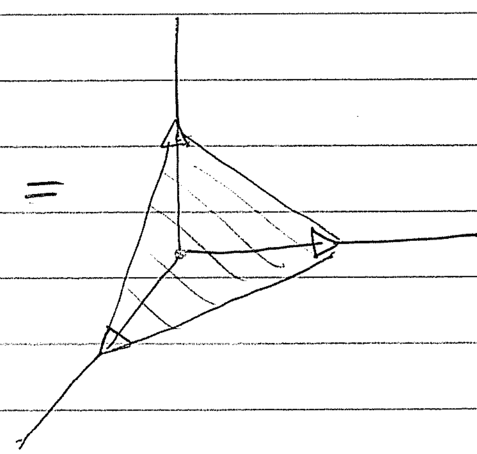
Recall: A cone is simplicial if it is generated by a linearly independent set.

Example: The standard simplex in $\mathbb{R}^n$ is the convex hull of the standard basis vectors

$$\Delta(n) = \{ a_1 e_1 + \cdots + a_n e_n : a_1, \ldots, a_n \geq 0 \text{ and } a_1 + \cdots + a_n = 1 \}$$

The standard orthant is the cone over this

$$\mathbb{R}_+ \langle e_1, \ldots, e_n \rangle = \{ a_1 e_1 + \cdots + a_n e_n : a_1, a_2, \ldots, a_n \geq 0 \}$$

$\Delta(3) =$ 

Now consider the cone generated by some basis $\alpha_1, \dots, \alpha_n \in \mathbb{R}^n$:

$$C = \mathbb{R}_+ \langle \alpha_1, \dots, \alpha_n \rangle$$

The dual (polyhedral) cone is

$$C^* = \{ u \in \mathbb{R}^n : u^t \alpha_i \leq 0 \quad \forall i \}$$

To understand C^* better we define the dual basis $\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*$ by

$$(\alpha_i^*)^t \alpha_j = -\delta_{ij} = \begin{cases} -1 & i = j \\ 0 & i \neq j \end{cases}.$$

In other words, if $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}$

Then α_i^* are the rows of $-A^{-1}$:

$$\underbrace{\begin{pmatrix} \alpha_1^{*t} \\ \vdots \\ \alpha_n^{*t} \end{pmatrix}}_{-A^{-1}} \underbrace{\begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}}_{-I}$$

Now for any $u \in \mathbb{R}^n$ we can write

$$u = a_1^* \alpha_1^* + \cdots + a_n^* \alpha_n^*$$

for some unique $a_1^*, \ldots, a_n^*$.
Note that $u^t \alpha_i = -a_i^*$.

Thus we have

$$\begin{aligned} u \in C^* &\iff u^t \alpha_i \leq 0 \quad \forall\, i \\ &\iff -a_i^* \leq 0 \quad \forall\, i \\ &\iff a_i^* \geq 0 \quad \forall\, i. \end{aligned}$$

We conclude that C^* is generated by the dual basis.

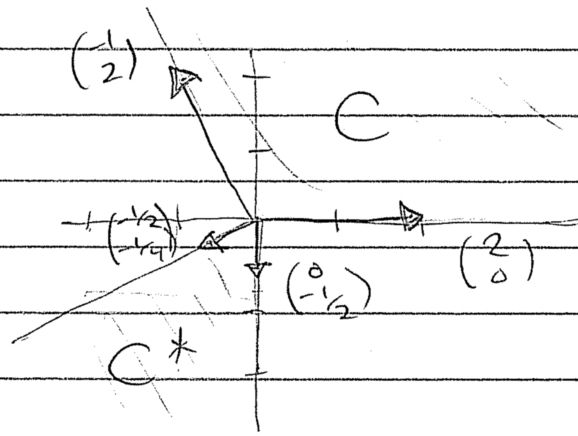
$$C^* = \mathbb{R}_+ \langle \alpha_1^*, \ldots, \alpha_n^* \rangle.$$

Furthermore, the dual of the dual basis is again $\alpha_1, \ldots, \alpha_n$. Hence

$$C^{**} = C.$$

$\blacksquare$

Example: Consider cone $C = \mathbb{R}_+\left\langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle$



Compute the dual basis.

$$-\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^{-1} = -\frac{1}{4}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

It is $\begin{pmatrix} -1/2 \\ -1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

$$\Longrightarrow \quad C^* = \mathbb{R}_+\left\langle \begin{pmatrix} -1/2 \\ -1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right\rangle$$

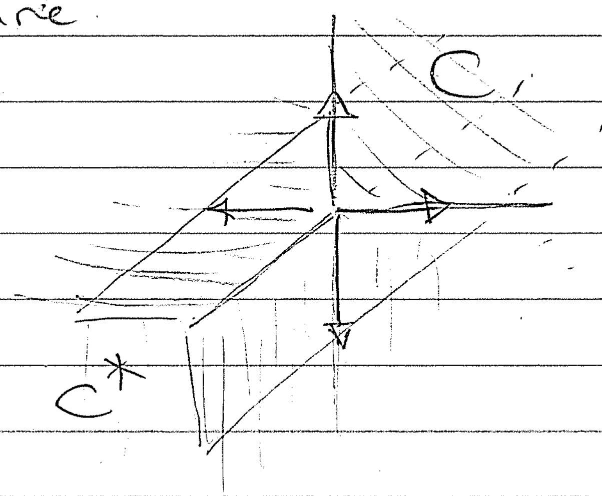
Exercise: Let C be the cone generated by linearly independent $\alpha_1, \dots, \alpha_m \in \mathbb{R}^n$ with $m < n$.

What can we say about $C^* \subseteq \mathbb{R}^n$?

Let $U = \mathbb{R}\langle \alpha_1, \dots, \alpha_m \rangle \subseteq \mathbb{R}^n$
and let C' be the dual cone of
C inside U.

Show that $C^* = C' + U^\perp \subseteq \mathbb{R}^n$

Picture



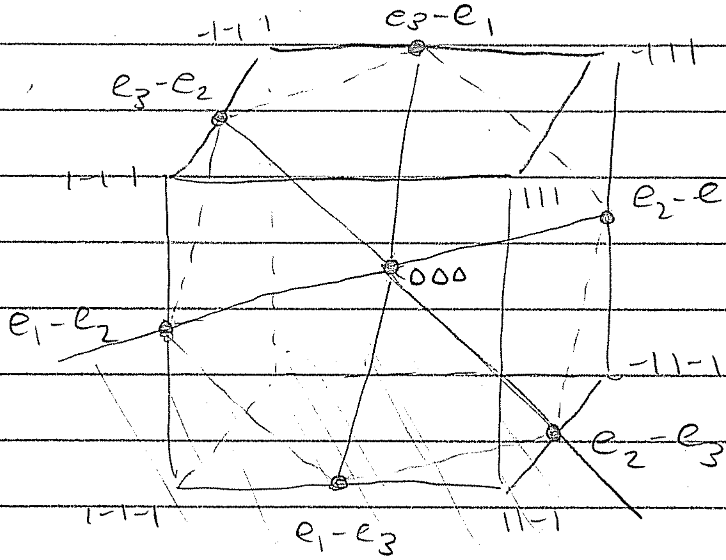
Big Example: My Favorite Cone.

Consider the cone in $\mathbb{R}^n$ generated by

$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \dots,\ \alpha_n = e_{n-1} - e_n.$

Actually it lives in the $(n-1)$-dim
hyperplane orthogonal to $e_1 + e_2 + \dots + e_n$.

Picture: Slice the cube $[-1,1]^3 \subseteq \mathbb{R}^3$ by the plane $(e_1+e_2+e_3)^\perp$

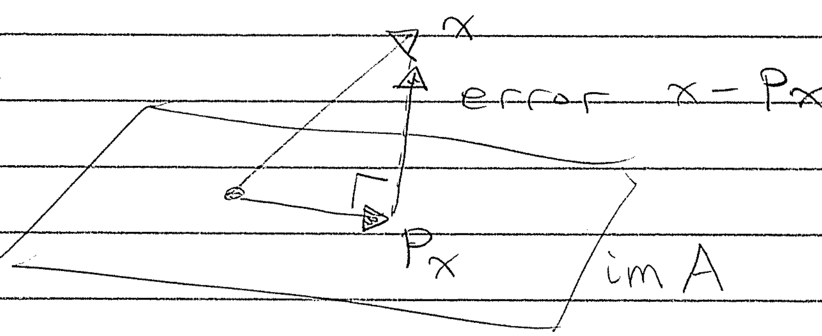


Better just project onto the space

$$\mathbb{R}^3_0 := \mathbb{R}\langle e_1-e_2, e_2-e_3\rangle = (e_1+e_2+e_3)^\perp$$

How? Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$

To project onto $\operatorname{im} A = \mathbb{R}^3_0$:



Given $x \in \mathbb{R}^n$ let $Px \in \mathbb{R}^n$, be the projection. Then

$$Px \in \operatorname{im} A \Rightarrow Px = A\hat{x} \text{ for } \underline{\text{some}}\ \hat{x} \in \mathbb{R}^n$$

Note that the error $x - Px$ is $\perp$ to im A. Hence

$$A^t(x - Px) = A^t(x - A\hat{x}) = 0.$$

$$A^t x - A^t A\hat{x} = 0$$

$$A^t x = A^t A \hat{x}.$$

$$(A^t A)^{-1} A^t x = \hat{x}$$

Finally

$$Px = A\hat{x} = A(A^tA)^{-1}A^t x.$$

$$\boxed{P = A(A^tA)^{-1}A^t}$$

Alternatively, $I-P$ projects onto the line $\begin{pmatrix}1\\1\\1\end{pmatrix}$ hence

$$I-P = \begin{pmatrix}1\\1\\1\end{pmatrix}\left[\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]^{-1}\begin{pmatrix}1&1&1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix}\left(\frac{1}{3}\right)\begin{pmatrix}1&1&1\end{pmatrix}.$$

$$= \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$\Longrightarrow P = \frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

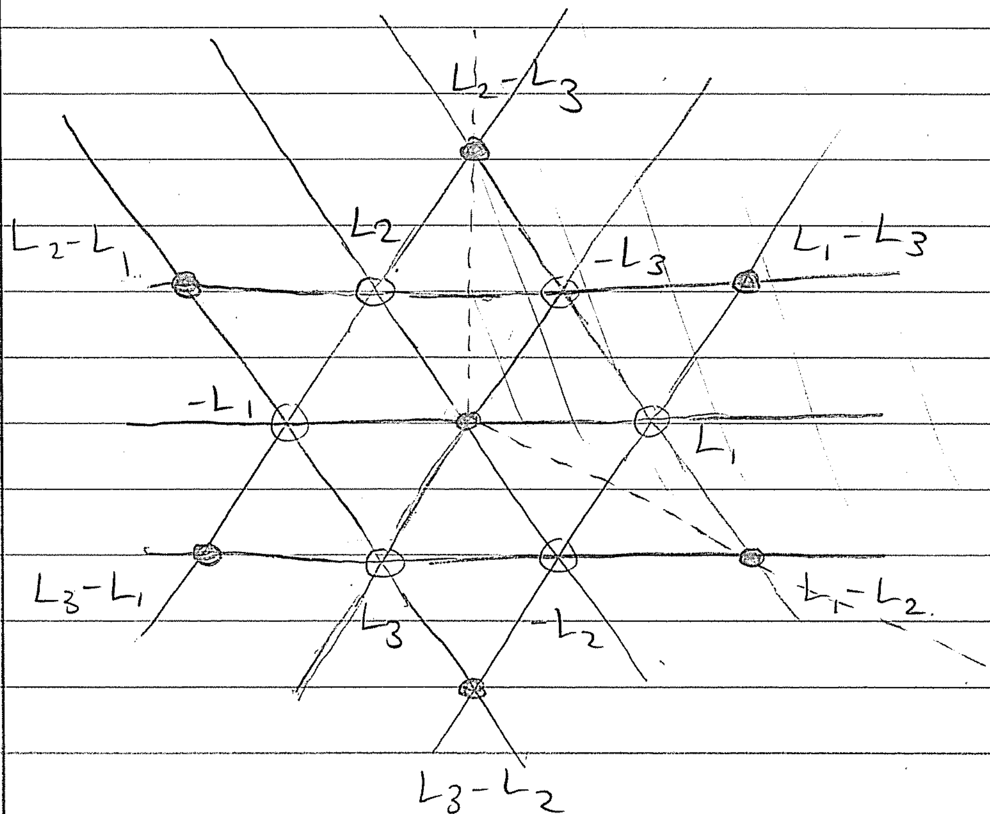
Let $L_i = Pe_i$, so

$$L_1 = \frac{1}{3}\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \quad L_2 = \frac{1}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}, \quad L_3 = \frac{1}{3}\begin{pmatrix}-1\\-1\\2\end{pmatrix}.$$

Note $L_1 - L_2 = e_1 - e_2$ because.

$$L_1 - L_2 = Pe_1 - Pe_2 = P(e_1 - e_2) = e_1 - e_2.$$

Picture of $\mathbb{R}^3_0$:



Note : $L_1 + L_2 = -L_3$ (strange).

Reason : $e_1 + e_2 = -e_3 + \underbrace{\cancel{(e_1 + e_2 + e_3)}}_{0}$.

Solid vertices = "root lattice" of type A_2

All vertices = "weight lattice" of type A_2