NO CLASS THURSDAY

Tues Feb 26

Consider any $G < \mathrm{Isom}(\mathbb{R}^n)$.

Lemma: If G has a finite orbit, say $e \in \mathbb{R}^n$ with

$$E = G \cdot e = \{ g(e) : g \in G \},$$

then G fixes a point: $\exists\, a \in \mathbb{R}^n$ with $g(a) = a \ \forall g \in G$.

Proof: For all $x \in \mathbb{R}^n$ set

$$m(x) := \max_{f \in E} d(x, f).$$

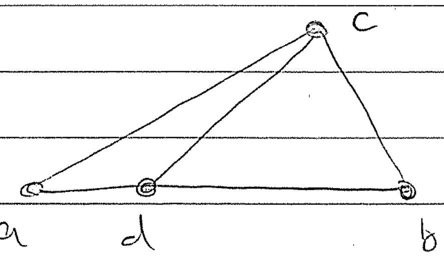
Now choose any $v \in \mathbb{R}^n$. The set

$$\{ x : d(x,f) \leq m(v) \text{ for all } f \in E \}$$

is closed and bounded, hence compact. Thus the continuous $m : \mathbb{R}^n \to \mathbb{R}$ attains a minimum on the set, which is also a minimum on all of $\mathbb{R}^n$.

Claim: The minimum is unique.

If m is minimized at a and b with $a \neq b$, choose d on the open segment (a, b). Then choose $c \in E$ with $m(d) = d(d,c)$



Then for one of a or b (say a) we have

$$m(d) = d(d,c) < d(a,c) \leq m(a)$$

contradicting minimality of $m(a)$.
Hence $a = b$. ///

Finally note that the action of G preserves the function $m : \mathbb{R}^n \to \mathbb{R}$.

If $m(a)$ is the unique minimum, we must have $m(g(a)) = m(a) \Rightarrow g(a) = a$ for all $g \in G$.

Hence a is a fixed point. ∎

[Remark: Proof also works for hyperbolic metric spaces.]

Now suppose that $G < \mathrm{Isom}(\mathbb{R}^n)$ is finite and generated by reflections

$$T = \{t_1, \ldots, t_N\} \subseteq G$$

Without loss assume that G fixes $0 \in \mathbb{R}^n$. Then the hyperplanes

$$\Sigma(G) = \{H_{t_1}, \ldots, H_{t_N}\}$$

are all <u>linear</u> (i.e. contain 0).

Conversely, suppose that the finite mirror system

$$\Sigma = \{H_1, \ldots, H_N\}$$

is closed (i.e. $t_{H_i}(H_j) \in \Sigma \ \forall i, j$). We saw last time that the group

$$G(\Sigma) = \langle t_H : H \in \Sigma \rangle$$

is finite. Hence G fixes a point (say 0) and we may assume that

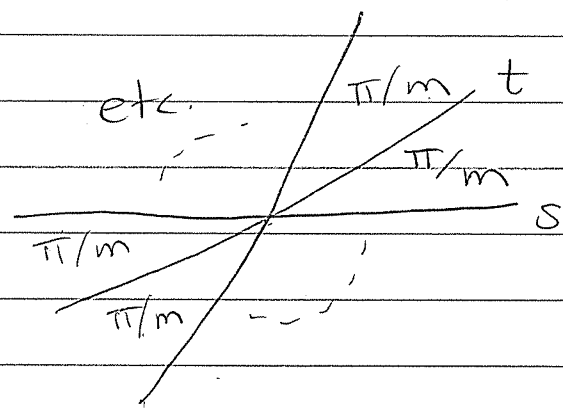
$H_1, H_2, \ldots, H_N$ are <u>linear</u>.

We get a bijective correspondence

Finite groups generated by reflections $\longleftrightarrow$ Finite closed systems of linear hyperplanes.

This allows us to classify all the FGGR of rank 2 (i.e. $< O(2)$).

The associated system of mirrors must have equal angles, hence



The corresponding FGGR is dihedral of order $2m$. We call this the Coxeter group of "type $G_2(m)$" (or "type $I_2(m)$")

Recall: It is generated by two adjacent reflections

$$G_2(m) = \langle s, t : s^2 = t^2 = (st)^m = 1 \rangle$$

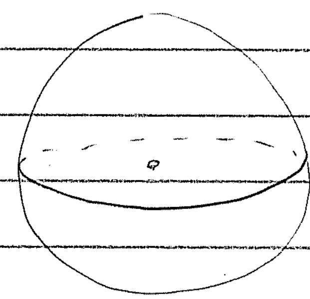
What about rank 3?

Consider a finite CMS Σ in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$.

Intersect with the unit sphere $S^2 \subseteq \mathbb{R}^3$ to get a tesselation of S^2 into isometric spherical n-gons for some $n \geq 1$.

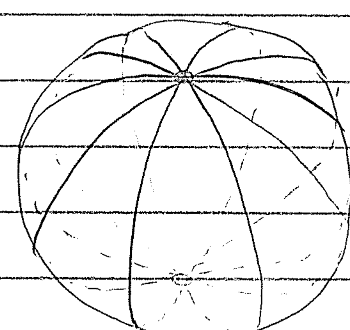
Examples:

$n = 1$



1-gons

$n = 2$



2-gon

Given an antipodal pair $\pm a \in S^2$, the planes $\{H \in \Sigma : \{\pm a\} \subseteq H\}$ form a dihedral sytem, with equal angles, all $\leq \frac{\pi}{2}$.

So Σ tesselates S^2 by isometric n-gons with all angles $\leq \pi/2$

Recall (Thomas Harriot, 1603):
Area of spherical triangle (radius 1) with angles α, β, γ is

$$\alpha + \beta + \gamma - \pi.$$

Corollary: Area of a spherical n-gon is

$$\text{angle sum} - (n-2)\pi.$$

Proof: Divide n-gon into $(n-2)$ spherical triangles. $\boxtimes$

Corollary: If all angles are $\leq \pi/2$ then $n \leq 3$.

Proof: We have

$$0 < \text{area} = \text{angle sum} - (n-2)\pi$$
$$\leq n \cdot \frac{\pi}{2} - (n-2)\pi$$

Hence $\frac{n}{2} - (n-2) > 0$.

$$n - 2n + 4 > 0$$

$$n < 4 \qquad \blacksquare$$

We conclude that Σ_i tesselates S^2 into isometric

(1) 1-gons $\Longrightarrow G(\Sigma_i) = \mathbb{Z}/2\mathbb{Z}$
"type A_1"

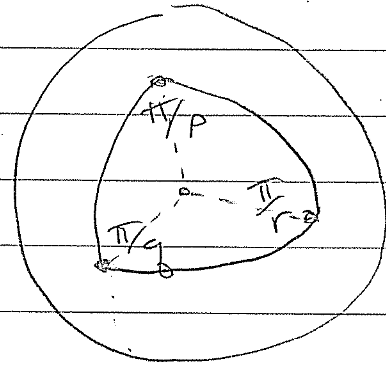
(2) 2-gons $\Longrightarrow G(\Sigma_i) =$ dihedral
"type $G_2(m)$"

OR

(3) 3-gons. In which case the angles must be $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ with

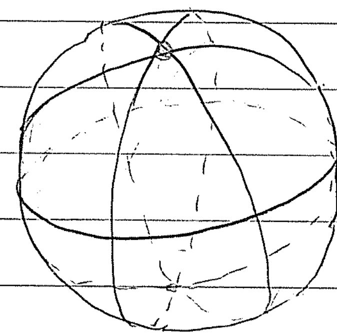
$$0 < \text{area} = \frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi.$$

$$\Longrightarrow 1 < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$



$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Cases: (p, q, r)



$(2, m, m)$
Group $G_2(m) \times A_1$

$(2, 3, 3)$ "type A_3"
Group $T \cup -(O \setminus T) \cong S_4$
Barycentric subdiv of tetrahedron

$(2, 3, 4)$ "type B_3"
Group $O \cup -O$
Barycentric subdiv of cube/octahedron

$(2, 3, 5)$ "type H_3"
Group $I \cup -I$
Barycentric subdiv of dodec/icosahedron

That's All.