Thurs Feb 14

Continued . . .

Decompose finite group G into conjugacy classes $G = C_1 \cup C_2 \cup \cdots \cup C_N$.

Let $\chi_1, \chi_2, \ldots, \chi_N$ be the irr. chars.
The "character table" has entries $(\chi_i(C_j))$

$$
\begin{array}{c|ccccc}
 & C_1 & \cdots & C_j & \cdots & C_N \\
\hline
\chi_1 & & & \vdots & & \\
\vdots & & & \vdots & & \\
\chi_i & \cdots & \cdots & \chi_i(C_j) & & \\
\vdots & & & & & \\
\chi_N & & & & &
\end{array}
$$

Recall: The irr. characters are orthonormal with respect to

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\varphi(g)}$$

Hence the rows of the char. table satisfy

$$\frac{1}{|G|} \sum_{k=1}^{N} \underbrace{|C_k|}_{} \chi_i(C_k) \overline{\chi_j(C_k)} = \langle \chi_i, \chi_j \rangle = \delta_{ij}$$

we don't like this!

So define the modified character table.

$$M = \left(\frac{\chi_i(C_j)}{\sqrt{|G|/|C_j|}} \right)_{i,j}$$

Then by definition the rows of M are orthonormal with respect to the standard form $(x, y) = \Sigma x_k \overline{y_k}$.

i.e. $M \overline{M}^t = I$.

It follows (from Rank-Nullity) that

$$\overline{M}^t M = I$$

hence the columns of M are orth.normal:

$$\sum_{k=1}^{N} \frac{\chi_k(C_i)}{\sqrt{|G|/|C_i|}} \cdot \overline{\frac{\chi_k(C_j)}{\sqrt{|G|/|C_j|}}} = \delta_{ij}$$

$$\Longrightarrow \sum_{k=1}^{N} \chi_k(C_i) \overline{\chi_k(C_j)} = \frac{|G|}{\sqrt{|C_i|}\sqrt{|C_j|}} \delta_{ij}$$

Columns of $(\chi_i(C_j))$ are orthogonal but not normal.

i.e. $\sum_{k=1}^{N} \chi_k(C_i)\overline{\chi_k(C_i)} = \frac{|G|}{|C_i|} = |\text{centralizer}|.$

Summary: The columns of the character table form an orthogonal (but not normal) basis for $\mathbb{C}^N$ with standard hermitian form.

Recall the binary polyhedral groups

$D^*_{p,q,r} < SU(2)$

and the affine Coxeter graphs

$T^*_{p,q,r}$ for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$.

The McKay Correspondence (1980 – ?):

The columns of the char table of $D^*_{p,q,r}$

$\|$

The eigenvectors of the (adj. matrix) of $T^*_{p,q,r}$.

Fun Corollary:

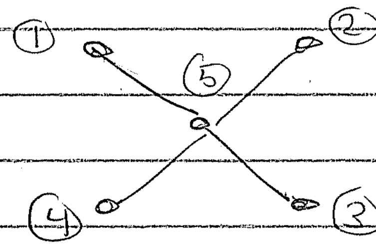
Let $(a_1, a_2, \ldots, a_N)$ be the "special" PF eigenvector for $T^*_{p,q,r}$. Then

$$\sum_{i=1}^{N} a_i^2 = |D^*_{p,q,r}| = \frac{4}{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1}$$

Small Example:

Affine type D_4 has graph

$T^*_{2,2,2}$ =



Adjacency matrix

$$\left(\begin{array}{cccc|c} & & & & 1 \\ & 0 & & & 1 \\ & & & & 1 \\ & & & & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

Spectrum:

$$\begin{array}{cl} 2 & (1, 1, 1, 1, 2) \\ -2 & (1, 1, 1, 1, -2) \\ \left.\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\{ & \begin{array}{l} (1, -1, -1, 1, 0) \\ (1, 1, -1, -1, 0) \\ (1, -1, 1, -1, 0) \end{array} \end{array}$$

not unique

The group $D^*_{2,2,2} = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ has char. table

	$\{1\}$	$\{-1\}$	$\{\pm k\}$	$\{\pm j\}$	$\{\pm i\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

"triality"
(not unique).

Check: $|Q_8| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$.

We have bijections

conj classes of $D^*_{p,q,r}$ $\longleftrightarrow$ eigenvectors of $T^*_{p,q,r}$

irreps of $D^*_{p,q,r}$ $\longleftrightarrow$ vertices of $T^*_{p,q,r}$.

Q: what about the edges of $T^*_{p,q,r}$?

Define the McKay graph of $D^*_{p,q,r} < SU(2)$:

Let χ be the defining (2-dim) char.

Let $\chi_1, \chi_2, \cdots, \chi_N$ be the irr. chars
(we know that $N = p+q+r-1$).

Tensor each irrep with χ and decompose

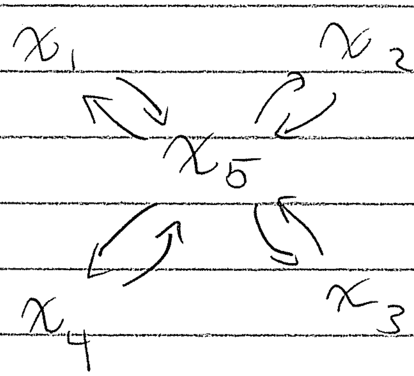
$$\chi \chi_i = \sum_{j=1}^{N} a_{ij} \chi_j$$

a_{ij}: some numbers $\in \mathbb{N}$

Define the McKay Graph $McK(D^*_{p,q,r})$

with vertices $1, 2, \ldots, N$
and with a_{ij} edges from $i \longrightarrow j$.

Example: for Q_8 , $\chi = \chi_5$



i.e. $x_5 x_1 = x_5$

$x_5 x_2 = x_5$

$x_5 x_3 = x_5$

$x_5 x_4 = x_5$

$x_5 x_5 = x_1 + x_2 + x_3 + x_4$

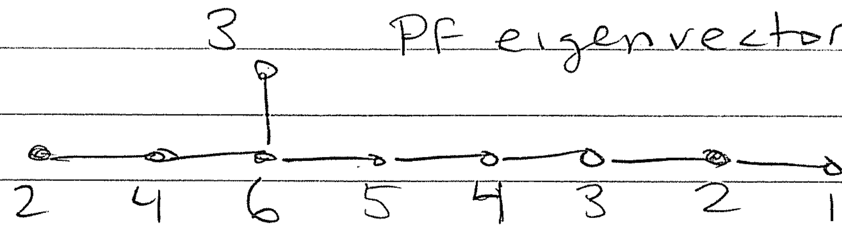
$$x_5^2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \checkmark$$

Bigger Example:

$D^*_{2,3,5} = I^* =$ binary icosahedral

$T^*_{2,3,5} = E_8^{(1)} =$ affine E_8.

PF eigenvector.



$1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 4^2 + 4^2 + 5^2 + 6^2$

$= 120 = |I^*|$.

Looking at char table (handout), we find the McKay graph

$$\begin{tikzcd}[column sep=small]
 & & \chi_5 \arrow[d, no head] & & & & \text{standard} & \\
\chi_3 \arrow[r, no head] & \chi_7 \arrow[r, no head] & \chi_9 \arrow[r, no head] & \chi_8 \arrow[r, no head] & \chi_6 \arrow[r, no head] & \chi_4 \arrow[r, no head] & \chi_2 \arrow[r, no head] & \chi_1 \\
 & & & & & & & \text{triv.}
\end{tikzcd}$$

Check:

$$\chi_2 \chi_7 = \chi_3 + \chi_9$$

χ_2	χ_7	$\chi_2\chi_7$		χ_3	+	χ_9
2	4	8		2		6
-2	4	-8	=	-2	+	-6
μ^+	-1	$-\mu^+$		μ^-		-1
$-\mu^-$	-1	μ^-		$-\mu^+$		1
μ^-	-1	$-\mu^-$		μ^+		-1
$-\mu^+$	-1	μ^+		$-\mu^-$		1
1	1	1		1		0
-1	1	-1		-1		0
0	0	0		0		0

Note: $\mu^+ + \mu^- = 1$, $\mu^+\mu^- = -1$.

The eigenvalues of $E_8^{(1)}$ are

$$\pm 2,\ \pm 1,\ 0,\ \pm \mu^+,\ \pm \mu^-$$

where $(x-\mu^+)(x-\mu^-) = x^2 - x - 1$

i.e. $\mu^\pm = \frac{1 \pm \sqrt{5}}{2}$

In general the eigenvector of $T^*_{p,q,r}$ corresponding to conj class C_i has eigenvalue $\chi(C_i) = \text{trace}(C_i)$

Recall the eigenvalues of the finite type graph $T_{p,q,r}$ are

$$2\cos\left(\frac{\pi(d_i - 1)}{h}\right)$$

where $d_1, \dots, d_{N-1} \in \mathbb{N}$ are special integers called the "degrees" and

$$h = \sum_{i=1}^{N} a_i$$

is the "Coxeter number".

To be continued...