University of Maryland College Park

Institute for Advanced Computer Studies TR–94–79
Department of Computer Science TR–3308

Perron–Frobenius Theory: A New Proof of the Basics*

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June 1994

ABSTRACT

This note presents a new proof of basic Perron–Frobenius theory of irreducible nonnegative matrices.

*This report is available by anonymous ftp from thales.cs.umd.edu in the directory pub/reports.

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ABSTRACT

This note presents a new proof of basic Perron–Frobenius theory of irreducible nonnegative matrices.

The purpose of this note is to provide new proofs of the basic facts about nonnegative irreducible matrices. The facts themselves are well known. If A is an irreducible nonnegative matrix then the spectral radius of A, written $\rho(A)$, is a simple eigenvalue of A corresponding to an positive eigenvector x. Moreover, $\rho(A)$ is a strictly increasing function of the elements of A. Perron [5] established these facts for positive matrices, and Frobenius [1] for nonnegative irreducible matrices. Frobenius went on to describe the structure of matrices for which there are other eigenvalues of magnitude $\rho(A)$, but that will not concern us here.

The proof given here uses a variant of the inverse power method to establish the existence of the Perron root $\rho(A)$ and the Perron vector x. To establish the simplicity and monotonicity of $\rho(A)$, it uses well known relations between right and left eigenvectors. We will take two facts as given.

Recall that a matrix A is reducible if there is a permutation matrix P such that

$$P^{\mathrm{T}}AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A is square. The first fact is that *if A is an irreducible, nonnegative matrix of order n, then $(I + A)^{n-1}$ is positive.*

The second fact concerns left and right eigenvectors. *Let λ be an eigenvalue of A and x a corresponding eigenvector. Then λ is multiple if and only if there is a left eigenvector corresponding to λ that is orthogonal to x.* This fact can be easily established from a Schur decompostion of A (e.g., see [4, §7.1.2]).

To establish the existence of a Perron vector we will use a variant of the inverse power method. Recall that the inverse power method approximates an eigenvector corresponding to an eigenvalue λ by choosing a scalar τ near λ and computing $x_\tau = (I - \tau A)^{-1}u$ for some u. If λ is simple and u is not inappropriately chosen, $x_\tau/\|x_\tau\|$ will approximate a normalized eigenvector corresponding to λ. The approximate becomes more accurate as τ approaches λ.

By replacing A with $A/\rho(A)$, we may assume that $\rho(A) = 1$. For $0 < \tau < 1$, let

$$x_\tau = (I - \tau A)^{-1}\mathbf{e} = \mathbf{e} + \tau A\mathbf{e} + \tau^2 A^2 \mathbf{e} + \cdots,$$

where $\mathbf{e}$ is the vector consisting entirely of ones. It is clear from the Neuman series on the right (which converges since $\rho(\tau A) < 1$) that $x \geq 0$.

We will now show that the the vectors x_τ become unbounded as $\tau \to 1$. Since $\rho(A^k) = 1$, we have $\|A^k\|_1 \geq 1$, where $\|\cdot\|_1$ is the usual column-sum norm. It follows that some element of A^k is not less than $1/n$. Since the 1-norm of x_τ is just the sum of all the elements of I, τA, $\tau^2 A^2$, ..., it follows that

$$\|x_\tau\|_1 \geq \frac{1/n}{1-\tau}.$$

Hence for any sequence $\tau_i \to 1$, we have

$$\|x_{\tau_i}\|_1 \to \infty.$$

The vectors $x_\tau/\|x_\tau\|_1$ lie on a closed and bounded set. Hence we can choose a sequence $\tau_i \to 1$ such that $\frac{x_{\tau_i}}{\|x_{\tau_i}\|} \to x \geq 0$. Then

$$0 = \lim_{i\to\infty} \frac{\mathbf{e}}{\|x_{\tau_i}\|_1} = \lim_{i\to\infty} \frac{(I - \tau_i A)x_{\tau_i}}{\|x_{\tau_i}\|_1} = (I - A)x.$$

Thus $x \geq 0$ is an eigenvector of A corresponding to $1 = \rho(A)$.

We will now show that any such x is positive. Since $Ax = x$, $(I + A)^{n-1}x = 2^{n-1}x$. Since $(I + A)^{n-1}$ is positive,

$$0 < (I + A)^{n-1}x = 2^{n-1}x.$$

To prove the simplicity and monotonicity of the Perron root, we will use a result of independent interest. *Let A be irreducible and nonnegative, and let $w \geq 0$ be nonzero. If there are scalars μ and ν such that*

$$\mu w \leq Aw \leq \nu w,$$

then

$$\mu \leq \rho(A) \leq \nu.$$

If the inequality $\mu w \leq Aw$ is strict in at least one component, then

$$\mu < \rho(A). \tag{1}$$

Likewise, if the inequality $Aw \le \nu w$ is strict in at least one component, then

$$\rho(A) < \nu. \tag{2}$$

We will establish the upper bounds in ν, the lower bounds being treated similarly. Note that along with A the matrix A^{T} is irreducible and nonnegative. Hence there is a positive vector y^{T} such that $y^{\mathrm{T}}A = \rho(A)y^{\mathrm{T}}$. Multiply the inequality $Aw \le \nu w$ by y^{T} to get

$$\rho(A)y^{\mathrm{T}}w = y^{\mathrm{T}}Aw \le \nu y^{\mathrm{T}}w.$$

The upper bound now follows on observing that $y^{\mathrm{T}}w > 0$. The strict inequality follows from the fact that if $Aw \le \nu w$ is strict in any component then $y^{\mathrm{T}}Aw < \nu y^{\mathrm{T}}w$.

To establish the simplicity of the Perron root, let x be the Perron vector of A. If $\rho(A)$ is not simple, there is a left eigenvector z^{T} corresponding to $\rho(A)$ such that z is orthogonal to x. Since x is positive, z^{T} must have both positive and negative components. Let $\hat{z}^{\mathrm{T}}$ be the vector obtained by setting all the negative components to zero. Since $z^{\mathrm{T}}A = \rho(A)z^{\mathrm{T}}$, it follows that $\hat{z}^{\mathrm{T}}A \ge \rho(A)\hat{z}^{\mathrm{T}}$. Now equality cannot hold in this relation, for otherwise $\hat{z}^{\mathrm{T}}$ would be a Perron vector and hence positive. Thus we have strict inequality in at least one component, and by (1) we have $\rho(A) > \rho(A)$—a contradiction.

Finally, we turn to the monotonicity of the spectral radius. Let $B \ge A$ with strict inequality in at least one component. Then B is nonnegative and irreducible. Let x be the Perron vector of A. Then

$$Bx \ge Ax = \rho(A)x,$$

with strict inequality in at least one component. Hence by (1), $\rho(B) > \rho(A)$.

There are three comments to be made about this proof. First, since $\rho(A)$ is a simple eigenvalue, it is easy to show that the vectors x_τ generated by the inverse power method converge to x as τ approaches one. Thus the existence proof is constructive.

Second, let λ be a simple eigenvalue with right eigenvector x and left eigenvector y, normalized so that $y^{\mathrm{T}}x = 1$. Then it is well known that the matrix of derivatives of λ with respect to the elements of its matrix is xy^{T}. In our case, $\lambda = \rho(A)$, and x and y are positive. Hence the matrix of derivatives is also positive. This provides another proof that the spectral radius of A increases with its elements, but one that predicts the rate of increase.

Finally the prominent role played by left eigenvectors in this development suggests that there is a duality at work here, a duality which would probably appear in sharper focus in the theory of matrices that are nonnegative with respect to a cone.

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