Thurs Sept 20.

PF is done! Now what?

Graph Spectra:

Let G be a simple, undirected graph.
Let A_G be the adjacency matrix

eg

$$\overset{1}{\circ} \text{———} \overset{2}{\circ} \text{———} \overset{3}{\circ} \qquad A_G = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

G

Think: $\circ \text{———} \circ \; = \; \circ \underset{1}{\overset{1}{\rightleftarrows}} \circ$

Then A_G is irreducible $\iff$ G is connected.
(and PF applies)

Some Observations:

- G undirected $\Rightarrow A_G = A_G^T$ ("symmetric")
  $\Rightarrow$ all eigenvalues of A_G are Real.

Proof: Suppose $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ and $0 \neq v \in \mathbb{C}^n$.

i.e. $$v^T \bar{v} = v_1 \bar{v}_1 + v_2 \bar{v}_2 + \cdots + v_n \bar{v}_n$$
$$= |v_1|^2 + \cdots + |v_n|^2$$
$$= \|v\|^2 \neq 0.$$

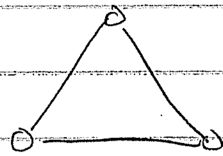
Then we have

$$\lambda \cancel{\|v\|^2} = \lambda (v^T \bar{v})$$
$$= (\lambda v)^T \bar{v}$$
$$= (Av)^T \bar{v}$$
$$= v^T A^T \bar{v}$$
$$= v^T \bar{A} \bar{v} \qquad (A^T = A = \bar{A})$$
$$= v^T (\overline{Av})$$
$$= v^T (\overline{\lambda v})$$
$$= v^T \bar{\lambda} \bar{v}$$
$$= \bar{\lambda} (v^T \bar{v})$$
$$= \bar{\lambda} \cancel{\|v\|^2}$$

$$\Longrightarrow \lambda = \bar{\lambda} \Longrightarrow \lambda \in \mathbb{R}. \qquad ///$$

Spectra of undirected graphs are Real.

eg. Consider the undirected 3-cycle.

$\tilde{A}_2 =$  $A_{\tilde{A}_2} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

The characteristic polynomial is

$$\chi(\tilde{A}_2, x) = \det \begin{pmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{pmatrix}$$

$$= x \det \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix} - (-1) \det \begin{pmatrix} -1 & -1 \\ -1 & x \end{pmatrix} + (-1) \det \begin{pmatrix} -1 & x \\ -1 & -1 \end{pmatrix}$$

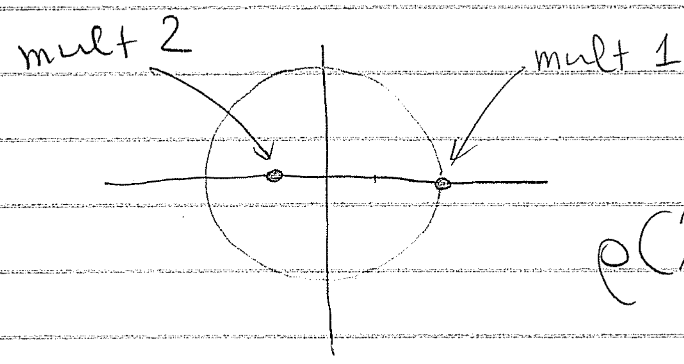
$$= x(x^2 - 1) + (-x - 1) - (1 + x)$$

$$= x^3 - 3x - 2$$

($x = -1$ is a root. Factor.)

$$= (x-2)(x+1)^2$$

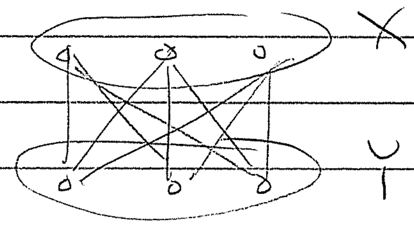
The spectrum is $-1, -1, 2$



$\rho(\tilde{A}_2) = 2$

- Say G is bipartite if $\exists$ bipartition of vertices $V = X \cup Y$ with no edges within X and no edges within Y

e.g. $K_{3,3} =$



Observation: If G is bipartite, its spectrum is symmetric about 0.

Proof: Since G is bipartite we write

$$A := A_G = \begin{array}{c} \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \begin{array}{l} \}\ |X| \\ \}\ |Y| \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{|X|} & \underbrace{\quad}_{|Y|} \end{array} \end{array}$$

[Actually $C = B^T$ but we don't need it.]

Suppose $Av = \lambda v$ and write

$$v = \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix}, \quad \begin{array}{l} v_1^T = (v_1, \dots, v_{|X|}) \\ v_2^T = (v_{|X|+1}, \dots, v_{|X|+|Y|}). \end{array}$$

So $\begin{pmatrix} \lambda v_1 \\ \hline \lambda v_2 \end{pmatrix} = \lambda v = Av = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix}$

$$= \begin{pmatrix} 0v_1 + Bv_2 \\ Cv_1 + 0v_2 \end{pmatrix} = \begin{pmatrix} Bv_2 \\ \hline Cv_1 \end{pmatrix}$$

$\Longrightarrow$ $Bv_2 = \lambda v_1$ & $Cv_1 = \lambda v_2$.

Now define $\tilde{v} := \begin{pmatrix} v_1 \\ \hline -v_2 \end{pmatrix}$. Then

$$A\tilde{v} = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \begin{pmatrix} v_1 \\ \hline -v_2 \end{pmatrix} = \begin{pmatrix} -Bv_2 \\ \hline Cv_1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda v_1 \\ \hline \lambda v_2 \end{pmatrix} = -\lambda \begin{pmatrix} v_1 \\ \hline -v_2 \end{pmatrix} = -\lambda \tilde{v}.$$

Hence λ is an evalue of A
$\Longleftrightarrow$ $-\lambda$ is an evalue of A.

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eg. Consider the 3-path.

$$A_3 = \circ\!-\!\!-\!\circ\!-\!\!-\!\circ \ , \quad A_{A_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

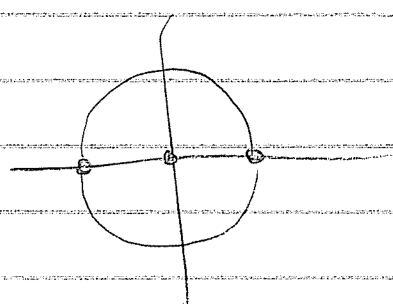
Charpoly:

$$\chi(A_3, x) = \det \begin{pmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{pmatrix}$$

$$= x \det \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix} - (-1) \det \begin{pmatrix} -1 & -1 \\ 0 & x \end{pmatrix} + 0.$$

$$= x(x^2 - 1) + (-x)$$
$$= x^3 - 2x$$
$$= x(x^2 - 2)$$

The spectrum is $\underbrace{-\sqrt{2},\ 0,\ \sqrt{2}}_{\text{symmetric}}$ ✓



$$\rho(A_3) = \sqrt{2} \approx 1.41$$

Theorem: Consider the n-path.

$$A_n = \underbrace{\circ - \circ - \circ \; \cdots \; \circ - \circ}_{n \text{ vertices}}$$

Its spectrum is

$$2\cos\left(\frac{\pi}{n+1}\right),\ 2\cos\left(\frac{2\pi}{n+1}\right),\ \ldots,\ 2\cos\left(\frac{n\pi}{n+1}\right).$$

<u>Proof</u>: The charpoly is

$$\chi(A_n, x) = \det \begin{pmatrix} x & -1 & & 0 \\ -1 & x & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & x \end{pmatrix}$$

$$= x \det \underbrace{\begin{pmatrix} x & -1 & & \\ -1 & x & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & x \end{pmatrix}}_{n-1} - (-1) \det \begin{pmatrix} -1 & -1 & & & \\ 0 & x & -1 & & \\ 0 & -1 & x & \ddots & -1 \\ & & & -1 & x \end{pmatrix}$$

$$= x\,\chi(A_{n-1}, x) + \left[(-1) \det \underbrace{\begin{pmatrix} x & -1 & & \\ -1 & x & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & x \end{pmatrix}}_{n-2} \right]$$

$$= x\,\chi(A_{n-1}, x) - \chi(A_{n-2}, x).$$

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"Three-term recurrence"

$$\boxed{\chi(A_n, x) = x\,\chi(A_{n-1}, x) - \chi(A_{n-2}, x).}$$

With initial conditions $\chi(A_1, x) = \det(x) = x$.

$$\chi(A_2, x) = \det\begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix} = x^2 - 1$$

Now what?

Recall for all angles $\alpha, \beta \in \mathbb{R}$ we have

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta.$$

"rotate by $\alpha+\beta$" "rotate by α then rotate by β"

In particular, for $\theta \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\sin(n\theta + \theta) = \sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta.$$

$$+ \quad \sin(n\theta - \theta) = \sin(n\theta)\cos\theta - \cos(n\theta)\sin\theta.$$

$$\sin((n+1)\theta) + \sin((n-1)\theta) = (2\cos\theta)\sin(n\theta).$$

$(*)$ $$\boxed{\sin((n+1)\theta) = (2\cos\theta)\sin(n\theta) - \sin((n-1)\theta).}$$

Hmm...

$\Longrightarrow$ We can recursively express $\sin(n\theta)$ as a function of $\cos\theta$.

DEF: Let $S_n(x) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$.

expressed as a function of $x := 2\cos\theta$.

Then (*) $\Longrightarrow$

$$\boxed{S_n(x) = x\,S_{n-1}(x) - S_{n-2}(x)}$$

Initial conditions ?

$$S_1(x) = \frac{\sin(2\theta)}{\sin\theta} = \frac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}} = 2\cos\theta = x. \checkmark$$

$$S_2(x) = \frac{\sin(3\theta)}{\sin\theta} = ?$$

de Moivre: $\cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\Longrightarrow \sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

$$\Longrightarrow S_2(x) = 3\cos^2\theta - \sin^2\theta$$
$$= 3\cos^2\theta - (1-\cos^2\theta)$$
$$= 4\cos^2\theta - 1$$
$$= (2\cos\theta)^2 - 1$$
$$= x^2 - 1 \quad \checkmark$$

Conclusion: $\chi(A_n, x) = S_n(x)$
as polynomials in x.
(this is (almost) the "Chebyshev polynomial of the 2nd kind"
$U_n(x) = S_n(2x)$.)

Finally, suppose $x = 2\cos\left(\frac{k\pi}{n+1}\right)$ for some $k \in [1, n]$, i.e. $\theta = k\pi/(n+1)$.

$$\chi(A_n, x) = S_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta} = \frac{\sin(k\pi)}{\underbrace{\sin(k\pi/(n+1))}_{\neq 0}}$$
$$= 0.$$

Since $\chi(A_n, x)$ has degree n we get

$$\chi(A_n, x) = \prod_{k=1}^{n}\left(x - 2\cos\left(\frac{k\pi}{n+1}\right)\right)$$

$\blacksquare$