Tue Sept 11

Note: Today $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Q: Given $T \in Mat(n, \mathbb{C})$, when does $\lim_{m \to \infty} T^m$ exist?

A: Consider Jordan Canonical form

$$T^m = C \begin{pmatrix} J_1^m & & & \\ & J_2^m & & \\ & & \ddots & \\ & & & J_k^m \end{pmatrix} C^{-1}$$

Then $\lim_{m \to \infty} T^m$ exists

$\iff \lim_{m \to \infty} J_i^m$ exists for all i.

Also recall that

$$J_i^m = \underbrace{\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}}_{b \times b}^m = \begin{pmatrix} \lambda_i^m & \binom{m}{1}\lambda_i^{m-1} & \cdots & \binom{m}{b-1}\lambda_i^{m-b+1} \\ & \lambda_i^m & & \vdots \\ & & \ddots & \binom{m}{1}\lambda_i^{m-1} \\ & & & \lambda_i^m \end{pmatrix}$$

There are two cases:

Case 1: If $b > 1$ then

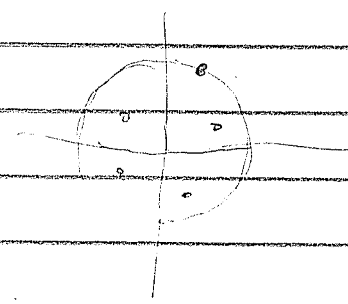
$$J_i^m \longrightarrow 0 \quad \text{if} \quad |\lambda_i| < 1$$

and diverges otherwise.

Case 2: If $b = 1$ then

$J_i^m \to (0)$ if $|\lambda_i| < 1$
$J_i^m \to (1)$ if $\lambda_i = 1$.
J_i^m diverges if $|\lambda_i| > 1$.
J_i^m is periodic if $\lambda_i = e^{2\pi i d/\ell}$
(some ℓ-th root of 1)

In any case, let



$$\rho(T) := \max_i \{ |\lambda_i| \}.$$

"Spectral radius of T"

Then

$$\lim_{m \to \infty} T^m \text{ exists} \Rightarrow \rho(T) \leq 1$$

Good News: The problem of Real non-negative matrices is completely understood

Notation:

- Given a matrix / vector T, write

$$T \geq 0 \quad \text{or} \quad T > 0$$

to mean that the entries are non-negative or positive

- $T \in \mathrm{Mat}(n, \mathbb{R})$ is called <u>primitive</u> if $\exists\, k$ with $T^k > 0$

- T is called <u>irreducible</u> if $\forall\, i, j$ $\exists\, k = k(i,j)$ with $(T^k)_{ij} > 0$.

Lemma: $T \geq 0$ is irreducible $\iff$ underlying directed graph is "strongly connected" (i.e. $\forall\, i, j$ $\exists$ directed path $i \to \cdots \to j$)

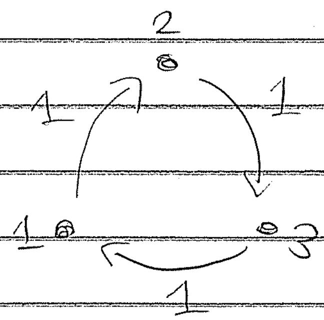
(Convention: $i \xrightarrow{w_{ij}} j$
If $w_{ij} = 0$ we say there is no edge from i to j.)

<u>Proof</u>: Recall

$$(T^k)_{ij} = \sum_{\substack{i \to j \text{ paths } p \\ \text{of length } k}} wt(p) \geq 0.$$

Inequality is strict $\iff$ $\exists$ some $i \to j$ path of length k

<u>eg</u>



$\longleftrightarrow \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

is irreducible because

$$T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $T^k = T^{k-3}$ for $k \geq 3$.

But T is NOT primitive.

Idea:

primitive = mixes well ("ergodic")
irreducible = maybe oscillates.

If T is NOT irreducible then $\exists$ permutation matrix Π such that

$$\Pi T \Pi^{-1} = \left(\begin{array}{c|c} A & C \\ \hline O & B \end{array}\right)$$

"reorder the vertices"

where A, B are square.

[Note: This case is not hopeless; see Andrew's grad seminar talk.]

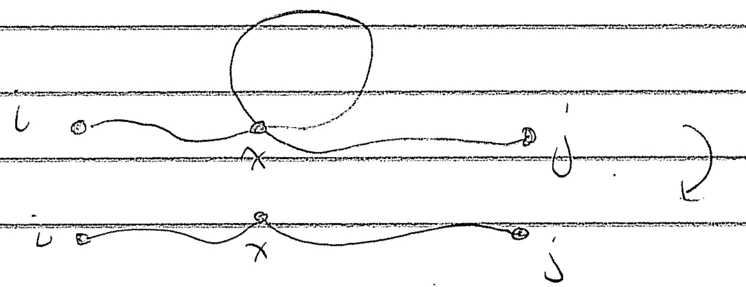
Lemma: If $T \geq 0$ is irreducible then $I + T$ is primitive.

Proof: suppose $T \in \mathrm{Mat}(n, \mathbb{R})$ (i.e. T is a strongly-connected weighted digraph on n vertices)

$\downarrow$

Then $\forall\, i,j \ \exists\ 0 \le k = k(i,j) \le n-1$ such that $(T^k)_{ij} > 0$.

(Proof: $\exists$ some $i \to j$ path. By removing any repeated vertices, it has length $\le n-1$



surgery.

)

It follows that

$$(I+T)^{n-1} = I + \binom{n-1}{1} T + \binom{n-1}{2} T^2 + \cdots + T^{n-1} > 0$$

$\blacksquare$

eg. $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \geq 0$

$I + T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \not> 0$

$(I+T)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} > 0$

Finally (!), here it is

Theorem (Perron 1907, Frobenius 1912).

Let $T \geq 0$ be irreducible. Then

(1) The spectral radius is an eigenvalue $\lambda_{max} = \rho(T) \in \mathbb{R}$.
(i.e. we have $|\lambda| \leq \lambda_{max}$ for all other e.values)

(2) λ_{max} has algebraic and geometric multiplicity 1 and a <u>positive</u> eigenvector $x > 0$.

(3) Any non-negative e.vector of T is a multiple of x.

(4) More generally, if $y \geq 0$, $y \neq 0$ is a vector and $\mu \in \mathbb{R}$ such that

$$Ty \leq \mu y$$

then $y > 0$ and $\mu \geq \lambda_{max}$.

(5) If $0 \leq S \leq T$, $S \neq T$, then every evalue σ of S satisfies $|\sigma| < \lambda_{max}$
(i.e. $\rho(S) < \rho(T)$)

(6) In particular, let $T_{(i)} = T$ with ith row and ith col deleted. Then $\rho(T_{(i)}) < \rho(T)$.

(7) If T is primitive then for all other evalues λ of T we have $|\lambda| < \lambda_{max}$.

Whew!

Immediate Corollary: If $T \geq 0$ is primitive then $\exists$ $C \in Mat(n, \mathbb{C})$ with

$$\left[\frac{1}{\rho(T)} T\right]^m = C \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & J_2^m & & \\ \vdots & & \ddots & \\ 0 & & & J_k^m \end{array}\right) C^{-1}$$

$$\longrightarrow C \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & O & \\ 0 & & & \end{array}\right) C^{-1}$$

$$= \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{pmatrix} \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & O & \\ 0 & & & \end{array}\right) \begin{pmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{pmatrix} = \vec{u}_1 \vec{v}_1^T$$

where $T\vec{u}_1 = \rho(T)\vec{u}_1$, $\vec{v}_1^T T = \rho(T)\vec{v}_1^T$ and $\vec{v}_1^T u_1 = 1$

Thurs Sept 13

Last time I set up the Perron-Frobenius Theorem. Today we will prove (the first part of) it.

(Follow along in Shlomo Sternberg's Chapter 9)

Let $n \times n$ matrix $T \geq 0$ be irreducible. Recall that $I+T$ is primitive. In fact,

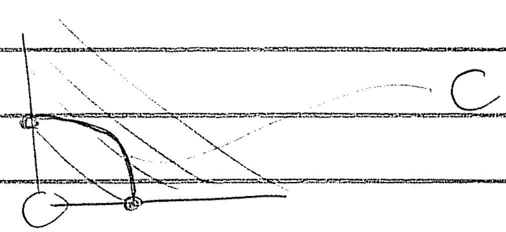
$$P := (I+T)^{n-1} > 0.$$

Note that

- $PT = TP$
- $u \leq v,\ u \neq v \Rightarrow Pu < Pv$

Also define the "positive orthant"

$$Q := \{ x \in \mathbb{R}^n : x \geq 0,\ x \neq 0 \}$$



and $C := \{ x \in Q : \|x\| = 1 \}$.

BEGIN PROOF:

Define $\forall\, z \in Q$

$$L(z) := \max\{s : sz \leq Tz\}$$
$$= \max\{s : sz_i \leq (Tz)_i \ \forall i\}$$
$$= \max\{s : s \leq \frac{(Tz)_i}{z_i} \ \forall z_i \neq 0\}$$
$$= \min\left\{\frac{(Tz)_i}{z_i} : z_i \neq 0\right\}$$

i.e. $L(z)$ = how much does T expand z?

Note $L(rz) = L(z)$ $\forall r > 0$, so L only depends on the "direction" of z.

If $sz \leq Tz$
Then $P(sz) \leq P(Tz)$.
$s(Pz) \leq T(Pz)$.

Hence $\{s : sz \leq Tz\} \subseteq \{s : s(Pz) \leq T(Pz)\}$
Taking "max" gives

$$L(z) \leq L(Pz).$$

By definition we have $L(z)z \leq Tz$.
But if $L(z)z \neq Tz$ then

$$P(L(z)z) < P(Tz)$$
$$L(z)(Pz) < T(Pz)$$
$$\Rightarrow L(z) < L(Pz).$$

Hypothetically

Hence $L(z) = L(Pz) \Rightarrow L(z)z = Tz$.
(i.e. z is an eigenvector with eigenvalue $L(z)$.

☆ Idea: Look for $z \in Q$ to maximize L. Then it must be an eigenvector.

Maximizing L:

Consider the image $P(C) = \{Pz : z \in C\}$.
Since P is continuous and $y \in P(C) \Rightarrow y > 0$, we have L continuous on $P(C)$.

Next, C compact $\Rightarrow P(C)$ compact.
Hence L attains a maximum value L_{max} on $P(C)$, say $L_{max} = L(x)$ with $x \in P(C)$.

Note that $L(Pz) \geq L(z) \Rightarrow L_{max}$ is the max of L on C also (and on Q since L only depends on direction).

Since $L(z) \leq L_{max}$ $\forall z \in Q$ and since $Px \in Q$ we have.

$$L(Px) \leq L_{max}.$$

Since $L_{max} = L(x) \leq L(Px)$ this implies that $L(Px) = L(x)$, which implies

$$Tx = L(x)\, x.$$

$$Tx = L_{max}\, x$$

and we can assume that $x > 0$ (else replace x by $Px > 0$).

Next issue: Show that every other eigenvalue λ satisfies $|\lambda| \leq L_{max}$

Let $Ty = \lambda y$. Goal: $|\lambda| \leq L_{max}$.

Define vector $|y|$ with i^{th} entry $|y_i|$, so $|y| \in Q$

Then $\lambda y_i = T y_i$

$\Rightarrow \lambda y_i = \sum_j T_{ij} y_j \quad \forall i$

$\Rightarrow |\lambda||y_i| \leq \sum |T_{ij}||y_j|$

$= \sum_j T_{ij} |y_j| \quad \forall i$

$\Rightarrow |\lambda||y| \leq T|y|$.

Thus by definition $|\lambda| \leq L(|y|) \leq L_{max}$ because $|y| \in Q$.

We conclude that $L_{max} = \rho(T)$ and we write $\lambda_{max} := L_{max}$.

Finally, note: If $\lambda_{max} = 0$ then $\rho(T) = 0 \Rightarrow T$ nilpotent (consider Jordan form) $\Rightarrow$ T is not irreducible (Why?).

Hence λ_{max}.

PF1 is proved.

Tues Sept 18

Today: Perron-Frobenius
(I'm not giving up yet!)

[Refer to G.W. Stewart's note:
"P-F Theory: a new proof of the basics"]

Theorem (P-F):
If $A \geq 0$ is irred then $\rho(A)$ is a simple evalue with pos evector $x > 0$
Moreover, if $0 \leq A \leq B$ with $A \neq B$ then $\rho(A) < \rho(B)$.

Proof: We already showed that $\rho(A)$ is an evalue with pos. evector $x > 0$.
Need to show

1. $\rho(A)$ is simple (multiplicity 1)
2. $0 \leq A \leq B$, $A \neq B \implies \rho(A) < \rho(B)$.
"spectral radius is a strictly increasing function"

Lemma: Consider irred $A \geq 0$, vector $w \geq 0$, $w \neq 0$ and scalars $\mu, \nu \in \mathbb{R}$.

If $\mu w \leq Aw \leq \nu w$ then $\mu \leq \rho(A) \leq \nu$.
Moreover,

$$\mu w \lneqq Aw \Rightarrow \mu < \rho(A)$$
$$Aw \lneqq \nu w \Rightarrow \rho(A) < \nu.$$

Proof of Lemma: Note that A^T is ≥ 0 and irreducible. Hence $\exists\ y > 0$ with $A^T y = \rho(A^T) y$. Transpose and apply x to get

$$y^T A = \rho(A^T) y^T$$
$$y^T A x = \rho(A^T) y^T x$$
$$y^T \rho(A) x = \rho(A^T) y^T x$$
$$\rho(A) \cancel{y^T x} = \rho(A^T) \cancel{y^T x}$$
$$\rho(A) = \rho(A^T)$$

~~or just observe~~ A and A^T have the same char. poly. hence the ~~same~~ spectrum

since $y^T x > 0$.
Now apply y^T to $Aw \leq \nu w$ to get

$$Aw \leq \nu w$$
$$y^T A w \leq \nu y^T w$$
$$\rho(A) \cancel{y^T w} \leq \nu \cancel{y^T w}$$
$$\rho(A) \leq \nu$$

since $y^T w > 0$.

If $Aw \lneq \nu w$ then since $y^T > 0$ we have

$$y^T A w < y^T \nu w$$
$$\Downarrow$$
$$\rho(A) < \nu$$

Lemma is done.

Proof of (1) ($\rho(A)$ is simple):
First note evalue λ with $Ax = \lambda x$ is multiple $\Leftrightarrow$ $\exists$ $y^T A = \lambda y^T$ with $y^T x = 0$. (Exercise).

Suppose for contradiction that $\rho(A)$ is multiple. Then $\exists$ $z^T A = \rho(A) z^T$ with $z^T x = 0$. Since $x > 0$, $z^T x = 0 \Rightarrow z$ has + and − entries. Let $\hat{z} = z$ with − entries set to 0. Note $\hat{z} \gneq 0$ and $\hat{z} \not> 0$. Then

$$z^T A = \rho(A) z^T \Rightarrow \hat{z}^T A \geq \rho(A) \hat{z}^T$$

Because:
- If $\hat{z}_i = 0$ then $(\hat{z}^T A)_i \geq 0 = \rho(A) \hat{z}_i$ ✓
- If $\hat{z}_i > 0$ then $\hat{z}_i = z_i$ hence

$$(\hat{z}^T A)_i = \hat{z}_1 a_{i1} + \cdots + \hat{z}_n a_{in}$$
$$\geq z_1 a_{i1} + \cdots + z_n a_{in}$$
$$= (z^T A)_i = \rho(A) z_i = \rho(A) \hat{z}_i$$

But if $\hat{z}^T A = \rho(A) \hat{z}^T$, then applying positive $(I+A)^{n-1} > 0$ to $\hat{z}^T \gneq 0$ gives

$$0 < \hat{z}^T (I+A)^{n-1} = \hat{z}^T (1+\rho(A))^{n-1}$$

$\Rightarrow \hat{z} > 0$. Contradiction.

Hence $\hat{z}^T A \gneq \rho(A) \hat{z}^T$ and the Lemma implies that $\rho(A) > \rho(A)$. Contradiction.

(1) is done. $\blacksquare$

Finally,
Proof of (2) $(0 \leq A \leq B,\ A \neq B \Rightarrow \rho(A) < \rho(B))$.

Note that A irred and $A \leq B \Rightarrow B$ irred.
(think strongly connected)

Then $B \gneq A \Rightarrow Bx \gneq Ax = \rho(A) x$
and the Lemma says $\rho(B) > \rho(A)$.

(2) is done. $\square$ PF is done $\blacksquare$