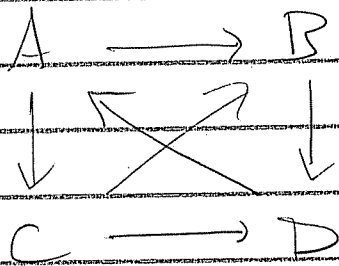


Thurs Aug 30

Today: A new beginning??

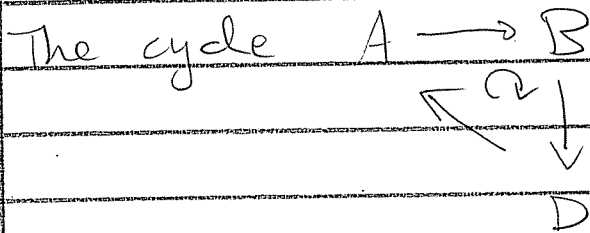
A round-robin tournament with  
4 players:



" $A \rightarrow B$ " means "A defeated B"

Question: Who won?

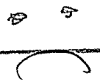
Problem: Rank the players fairly!



means any ranking has an upset

e.g.  $A > B > D$

But D defeated A. He is upset

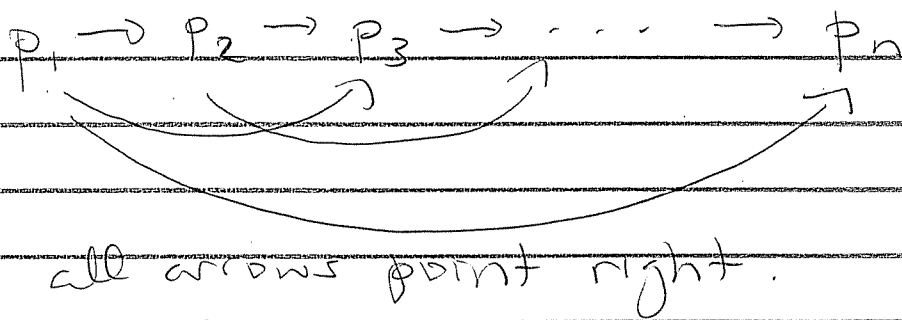


Option 1: Hope for no cycles.

# possible outcomes =  $2^{\binom{n}{2}}$  ← orient each edge.

# acyclic (or transitive) outcomes =  $n!$

Proof: There are  $n!$  ways to rank the players.  
Each ranking comes from a unique outcome



all arrows point right.



[Different Language:

"tournament with  $n$  players" = "orientation of the complete graph  $K_n$ "

Given graph  $G$  with  $n$  vertices, define

$\chi_G(k) = \#$  proper vertex colorings using at most  $k$  colors.

(it's a polynomial in  $k$  of degree  $n$ .  
The "chromatic polynomial")

e.g. let  $C_n = \overbrace{\circ \rightarrow \circ \rightarrow \dots \rightarrow \circ}^{n \text{ vertices}}$

$$\text{then } \chi_{C_n}(k) = k(k-1)^{n-1}$$

$$\text{e.g. } \chi_{K_n}(k) = k(k-1)(k-2)\dots(k-(n-1))$$

Theorem (Stanley, 1972)

$$(-1)^n \chi_G(-1) = \# \text{ acyclic orientations of } G$$

Hence,

$$\begin{aligned} \# \text{ acyclic tournaments} &= (-1)^n \chi_{K_n}(-1) \\ &= (-1)^n (-1)(-2)\dots(-n) \\ &= n! \quad \checkmark \end{aligned}$$

In any case,

Option 1 FAILS because

$$\frac{\binom{n}{2}}{n!} \rightarrow 0 \quad \text{really fast}$$

Option 2: Minimize upsets. X

- computationally infeasible
- inelegant
- not unique! (not fair)

Option 3: Use the upsets.

Idea: Beating a good player is better than beating a bad player.

Possible Issue: Who's "good" if we didn't rank them yet? (seems circular...)

Try anyway:

Initialize: For each player define a "score"

$w_i(\text{player } P) = \# \text{ games } P \text{ won.}$

$$= \sum_{\substack{X \\ P \rightarrow X}} 1$$

= # out-arrows  
( "out-degree" )

A B C D

Score vector:  $\vec{w}_1 = (2, 1, 2, 1)$

Do it again: Define a "new score"

$$w_2(P) = \sum_{\substack{X \\ P \rightarrow X}} w_1(X).$$

$$= \sum_{\substack{X \\ P \rightarrow X}} \sum_{\substack{Y \\ X \rightarrow Y}} 1$$

= paths of length 2  
starting at P

A B C D

So  $\vec{w}_2 = (3, 1, 2, 2)$

Repeat: For  $n \geq 1$  define.

$$w_n(P) := \sum_{\substack{X \\ P \rightarrow X}} w_{n-1}(X)$$

= # paths length n from P.

and  $w_0(P) := 1 \quad \forall P$ .

This is a matrix equation

Adjacency matrix :

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D
 \end{array}
 \begin{pmatrix}
 & A & B & C & D \\
 A & 0 & 1 & 1 & 0 \\
 B & 0 & 0 & 0 & 1 \\
 C & 0 & 1 & 0 & 1 \\
 D & 1 & 0 & 0 & 0
 \end{pmatrix}
 \quad \& \quad
 \vec{w}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Then  $\vec{w}_n = M \vec{w}_{n-1}$

$$\vec{w}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

just the row sums

$$\vec{w}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

etc.

$$\vec{w}_{10} = \begin{pmatrix} 35 \\ 21 \\ 34 \\ 26 \end{pmatrix} \begin{matrix} 1 \\ 4 \\ 2 \\ 3 \end{matrix}
 \quad
 \vec{w}_{20} = \begin{pmatrix} 1037 \\ 547 \\ 933 \\ 731 \end{pmatrix} \begin{matrix} 1 \\ 4 \\ 2 \\ 3 \end{matrix}$$

can we stop?

We get  $\vec{w}_n = M^n \vec{w}_0$  and we really want

$$\vec{w}_\infty = \lim_{n \rightarrow \infty} M^n \vec{w}_0$$

Small Problem: It blows up!

But we can fix it.

Normalized Recurrence:

$$\vec{w}_n := \frac{M \vec{w}_{n-1}}{\|M \vec{w}_{n-1}\|} \quad \text{a unit vector}$$

Suppose for the moment that

$\vec{w}_\infty := \lim_{n \rightarrow \infty} \vec{w}_n$  exists. Then we have

$$\vec{w}_\infty = \vec{w}_{\infty+1} = \frac{M \vec{w}_\infty}{\|M \vec{w}_\infty\|}$$

$$\Rightarrow M \vec{w}_\infty = \|M \vec{w}_\infty\| \vec{w}_\infty$$

eigenvalue

eigenvector

$$\approx 1.395$$

$$\approx \begin{pmatrix} 0.625 \\ 0.321 \\ 0.552 \\ 0.448 \end{pmatrix}$$



Q: So does  $\vec{w}_\infty$  exist?

Short Answer: Yes, and in fact it's (mostly) independent of the initial condition  $\vec{w}_0$ .

Longer Answer ...

Hypothesis ("Perron-Frobenius Theorem"):

$M$  has a real eigenvalue  $\lambda > 0$  such that

- $|\sigma| < \lambda$  for all eigenvalues  $\sigma \neq \lambda$
- $\lambda$  has a positive eigenvector  $\vec{v}$
- $\lambda$  has multiplicity 1
- If  $\vec{u}$  is a positive e.vector for  $M$  then  $\vec{u}, \vec{v}$  are proportional.

∴

If this hypothesis holds then  $\vec{w}_\infty$  exists.

I.O.U.