Thurs Aug 30

Today: A new beginning. ??

A round-robin tournament with 4 players:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur] & D \arrow[ul] \end{tikzcd}$$

"$A \to B$" means "A defeated B"

Question: Who won?
Problem: Rank the players fairly!

The cycle
$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ & D \arrow[ul] \end{tikzcd}$$

means any ranking has an upset

e.g. $A > B > D$

But D defeated A. He is upset

Option 1: Hope for no cycles.

$\#$ possible outcomes $= 2^{\binom{n}{2}}$ ← orient each edge.

$\#$ acyclic (or transitive) outcomes $= n!$

Proof: There are $n!$ ways to rank the players. Each ranking comes from a unique outcome

$$p_1 \to p_2 \to p_3 \to \cdots \to p_n$$

all arrows point right. $\square$

[Different Language:
"tournament with n players" = "orientation of the complete graph K_n"

Given graph G with n vertices, define

$\chi_G(k) = \#$ proper vertex colorings using at most k colors.

(it's a polynomial in k of degree n.
The "chromatic polynomial")

e.g. let $C_n = \overbrace{\circ - \circ - \circ \cdots \circ - \circ}^{n \text{ vertices}}$

Then $\chi_{C_n}(k) = k(k-1)^{n-1}$.

e.g. $\chi_{K_n}(k) = k(k-1)(k-2)\cdots(k-(n-1))$

Theorem (Stanley, 1972).

$(-1)^n \chi_G(-1) = \#$ acyclic orientations of G.

Hence,

$$\begin{aligned}\#\text{ acyclic tournaments} &= (-1)^n \chi_{K_n}(-1) \\ &= (-1)^n(-1)(-2)\cdots(-n) \\ &= n! \quad \checkmark\end{aligned}$$

In any case,

Option 1 FAILS because

$$\frac{2^{\binom{n}{2}}}{n!} \longrightarrow 0 \quad \text{really fast}$$

Option 2 : Minimize upsets.
- computationally infeasible
- inelegant
- not unique! (not fair)

Option 3 : Use the upsets.

Idea: Beating a good player is better than beating a bad player.

Possible Issue: Who's "good" if we didn't rank them yet? (seems circular...)

Try anyway:

Initialize: For each player define a "score"

$$w_1(\text{player } P) = \#\text{ games } P \text{ won.}$$

$$= \sum_{\substack{X \\ P \to X}} 1$$

$$= \#\text{ out-arrows}$$
("out-degree")

$$\begin{array}{ccccc} & A & B & C & D \end{array}$$

Score vector : $\vec{w}_1 = (2, 1, 2, 1)$

Do it again : Define a "new score"

$$w_2(P) = \sum_{\substack{X \\ P \to X}} w_1(X).$$

$$= \sum_{\substack{X \\ P \to X}} \sum_{\substack{Y \\ X \to Y}} 1$$

= paths of length 2 starting at P.

$$\begin{array}{ccccc} & A & B & C & D \end{array}$$

So $\vec{w}_2 = (3, 1, 2, 2)$

Repeat : For $n \geq 1$ define

$$w_n(P) := \sum_{\substack{X \\ P \to X}} w_{n-1}(X)$$

= # paths length n from P.

and $w_0(P) := 1 \quad \forall P$.

This is a matrix equation

Adjacency matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \& \quad \vec{w}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Then $\vec{w}_n = M \vec{w}_{n-1}$

$$\vec{w}_1 = \underbrace{\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{w}_0} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

just the row sums.

$$\vec{w}_2 = \underbrace{\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}}_{\vec{w}_1} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

etc. $\vec{w}_{10} = \begin{pmatrix} 35 \\ 21 \\ 34 \\ 26 \end{pmatrix} \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array} \quad \vec{w}_{20} = \begin{pmatrix} 1037 \\ 547 \\ 933 \\ 731 \end{pmatrix} \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array}$

can we stop?

We get $\vec{w}_n = M^n \vec{w}_0$ and we really want

$$\text{“}\vec{w}_\infty\text{''} = \lim_{n \to \infty} M^n \vec{w}_0$$

Small Problem: It blows up!
But we can fix it.

Normalized Recurrence:

$$\vec{w}_n := \frac{M\vec{w}_{n-1}}{\|M\vec{w}_{n-1}\|} \quad \text{a unit vector}$$

Suppose for the moment that
$\vec{w}_\infty := \lim_{n \to \infty} \vec{w}_n$ exists. Then we have

$$\vec{w}_\infty = \text{“}\vec{w}_{\infty+1}\text{''} = \frac{M\vec{w}_\infty}{\|M\vec{w}_\infty\|}$$

$$\Longrightarrow \quad M\vec{w}_\infty = \underbrace{\|M\vec{w}_\infty\|}_{\substack{\text{eigenvalue} \\ \approx 1.395}} \; \underbrace{\vec{w}_\infty}_{\text{eigenvector}}$$

eigenvector $\approx \begin{pmatrix} 0.625 \\ 0.321 \\ 0.552 \\ 0.448 \end{pmatrix}$

Q: So *does* $\vec{w}_\infty$ exist?

Short Answer: Yes, and in fact it's (mostly) independent of the initial condition $\vec{w}_0$.

Longer Answer...

Hypothesis ("Perron-Frobenius Theorem"):

M has a real eigenvalue $\lambda > 0$ such that

- $|\sigma| < \lambda$ for all eigenvalues $\sigma \neq \lambda$
- λ has a positive eigenvector $\vec{v}$
- λ has multiplicity 1
- If $\vec{u}$ is a positive e.vector for M then $\vec{u}, \vec{v}$ are proportional.

If this hypothesis holds then $\vec{w}_\infty$ exists.

I.O.U.