Thurs Nov 29

Last day of the semester. ☹

Recall:

$$\mathrm{Isom}(AV, Q) = V \rtimes O(V, Q)$$

and $\forall\ \varphi \in \mathrm{Isom}(AV, Q)$ $\exists$ (affine) reflections

$$\varphi = R_1 \circ R_2 \circ \cdots \circ R_k$$

where $k \leq \dim(V) + 1$

Today: finite groups of isometries.

Fixed Point Theorem:
If $G \leq \mathrm{Isom}(AV, Q)$ is finite, then G fixes a point, i.e. G is isomorphic to a subgroup of $O(V, Q)$.

(This is easy, but the proof is nice.)

Define the full affine group

$$\mathrm{Aff}(V) := V \rtimes GL(V)$$

Lemma: <u>Affine</u> maps preserve <u>affine</u> combinations. i.e. $\forall\ \varphi \in \mathrm{Aff}(V)$, vectors $x_1, \dots, x_n \in V$ and scalars $c_1, \dots, c_n \in \mathbb{F}$ with $c_1 + \cdots + c_n = 1$, have

$$\varphi(c_1 x_1 + \cdots + c_n x_n) = c_1 \varphi(x_1) + \cdots + c_n \varphi(x_n).$$

<u>Proof</u>: Let $\varphi = t_\alpha \circ A \in V \rtimes GL(V)$.

Then $t_\alpha \circ A(c_1 x_1 + \cdots + c_n x_n)$

$$= t_\alpha(c_1 A x_1 + \cdots + c_n A x_n)$$

$$= (c_1 A x_1 + \cdots + c_n A x_n) + \alpha.$$

$$= (c_1 A x_1 + \cdots + c_n A x_n) + (c_1 \alpha + \cdots + c_n \alpha)$$

$$= c_1(A x_1 + \alpha) + \cdots + c_n(A x_n + \alpha).$$

$$= c_1[t_\alpha \circ A(x_1)] + \cdots + c_n[t_\alpha \circ A(x_n)]$$

$$= c_1 \varphi(x_1) + \cdots + c_n \varphi(x_n)$$

$\blacksquare$

Could have <u>defined</u> affine maps this way.

Proof of Fixed Point Theorem:

Let $G \leq \mathrm{Isom}(AV, Q)$ be finite and let $x \in AV$ be any point. Consider the G-orbit

$$G(x) := \{ g(x) \in AV : g \in G \}$$

Suppose $G(x) = \{x_1, x_2, \ldots, x_n\}$ (in particular, $n \mid |G|$) and consider the centroid

$$x' = \frac{1}{n}(x_1 + \cdots + x_n)$$

(Let's assume $|G|$ and $\mathrm{char}\, \mathbb{F}$ are coprime)

Now let $\varphi \in G$. Since $\mathrm{Isom}(AV, Q) \leq \mathrm{Aff}(V)$, the Lemma says

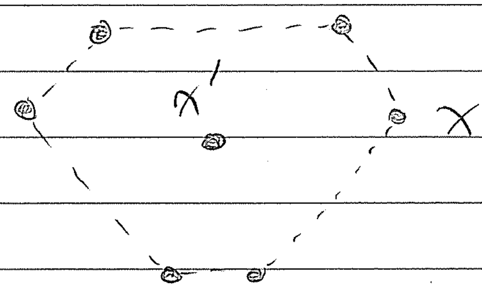
$$\varphi(x') = \frac{1}{n}\varphi(x_1) + \cdots + \frac{1}{n}\varphi(x_n)$$

$$= \frac{1}{n}(\varphi(x_1) + \cdots + \varphi(x_n)).$$

But note $\{\varphi(x_1), \ldots, \varphi(x_n)\} = \{x_1, \ldots, x_n\}$

$\Longrightarrow \varphi(x') = x'$ ∎

Picture.



Idea: If $|G(x)| = |G|$ then $G(x)$ is a geometric relalization of G, with possibly nice structure (polytope, etc.)

Finally, back to topological fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$

DEF: We say G is a topological group if

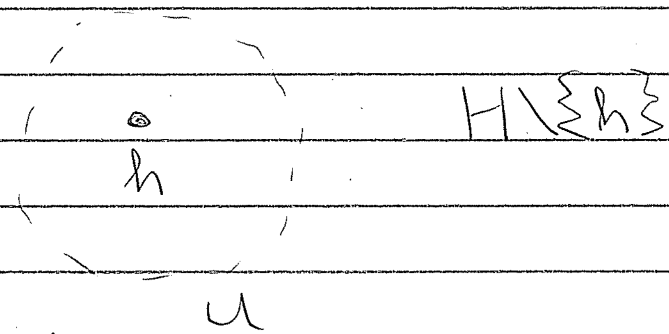
1. G is a group
2. G is a topological space
3. The maps

- $\mu : G \times G \to G$, $\mu(g, h) = gh$
- $inv : G \to G$, $inv(g) = g^{-1}$

are continuous.

(Short: G is a group object in the category of topological spaces)

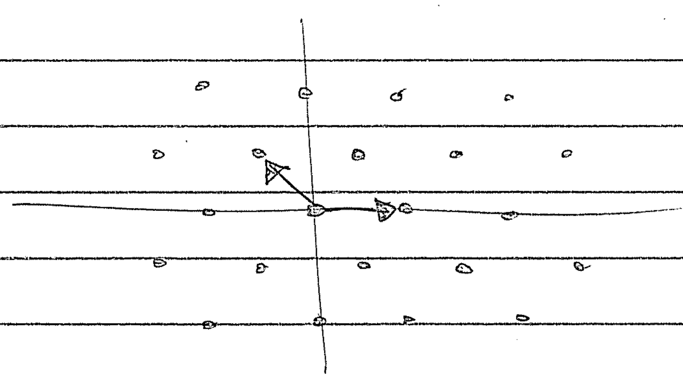
Say subgroup $H \leq G$ is discrete if the relative topology is discrete. i.e. each $h \in H$ is isolated in G. ($\exists$ open G-nbhd $h \in U$ with $U \cap H = \{h\}$)



Example: $(\mathbb{R}^n, +)$ is topological in the usual way.

A discrete subgroup of $(\mathbb{R}^n, +)$ is called a "lattice".

Picture $\mathbb{R}^2$.



Nontrivial (!) Theorem:
Every lattice $\Lambda \leq \mathbb{R}^n$ has a basis.
i.e. $\exists$ $\mathbb{R}$-linearly independent
$\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}^n$ $(k \leq n)$ with

$$\mathbb{Z}^k \approx \Lambda = \{ \sum_i c_i \alpha_i : c_i \in \mathbb{Z} \} \leq \mathbb{R}^n$$

This k is called the "rank" of Λ.
(We reserve the word "dimension" for vector spaces over fields).

Observe: if $\Lambda \leq \mathbb{R}^n$ is full-rank then $\mathbb{R}^n / \Lambda$ is a "torus".

Now consider

$\mathbb{A}\mathbb{R}^n$ = Euclidean space
$\mathrm{Isom}(\mathbb{A}\mathbb{R}^n)$ = Euclidean geometry.
(in the sense of Klein).

If $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$ is <u>discrete</u>,
one can show that

$$\Gamma = \Lambda \rtimes G.$$

where G is a <u>finite</u> subgroup of
$O(n)$, stabilizing a lattice $\Lambda \leq \mathbb{R}^n$.

i.e. $\forall g \in G, \alpha \in \Lambda$ we have $g(\alpha) \in \Lambda$

[Key step: $O(n)$ is compact, hence
$G \leq O(n)$ discrete $\Rightarrow$ G finite]

Sometimes discrete $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$
are called crystallographic groups.

Naive Goal:
Classify crystallographic groups

Case $n = 2$

There are 17, called the
"wallpaper groups" (M. Artin pg. 174)

and 7 "frieze groups"

Case $n=3$ (1892 Fyodorov-Schönflies)

There are 230. (Ugh.)

Hilbert's 18th problem (1900):

Are there $< \infty$ groups for each n?

Theorem (Bieberbach 1910): Yes.

However.....

Classify discrete $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$

Classify lattices $\Lambda \leq \mathbb{R}^n$

(Too hard.)

Classify finite $G \leq O(n)$.

(Impossible.)

Oh Well.

[Remark: There is one very special case.
The finite subgroups of $SO(3)$ are

(1) cyclic A_ℓ
(2) dihedral D_ℓ
(3) rotations of tetrahedron E_6
(4) rotations of cube/octahedron E_7
(5) rotations of icos./dodecahedron E_8

I.O.U. What this means
(McKay correspondence)]

To go further, we need some natural restriction.

Idea: Study discrete $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$ generated by reflections

BINGO!

THE END?