Thurs Nov 15.

Recall: We showed that

$$\mathrm{Isom}(AV,Q) = \underset{\curvearrowleft}{V \rtimes O(V,Q)}$$

natural action.

Matrix Representation:

$$\mathrm{Isom}(AV,Q) = \left\{ \left(\begin{array}{c|c} A & \alpha \\ \hline 0\cdots0 & 1 \end{array}\right) : \begin{array}{l} A \in O(V,Q) \\ \alpha \in V \end{array} \right\}$$

Current Goal: Prove the

Cartan - Dieudonné Theorem:
<1948 1948

If $\dim V = n$ then every isometry $\varphi \in \mathrm{Isom}(AV,Q)$ can be written as a product of $\leq n+1$ reflections.

Q: What is a "reflection"?

Recall: If $Q : V \to \mathbb{F}$ is non-degenerate and $W \subseteq V$ is any subspace, then

① $\dim V = \dim W + \dim W^{\perp}$
② $(W^{\perp})^{\perp} = W$.

If, moreover, $Q : W \to \mathbb{F}$ is non-degenerate (i.e. if $W \cap W^{\perp} = 0$; think: W avoids the "light cone") then we have

$$V = W \perp W^{\perp}$$

$\uparrow$ orthog. direct sum.

In this case we can define the "orthogonal projection" $P_W : V \to V$ by

$$P_W(w + u) := w$$

for all $w \in W$ and $u \in W^{\perp}$.
Note $\operatorname{im} P_W = W$ and $\ker P_W = W^{\perp}$.

Matrix:

$$[P_W] = \begin{array}{cc} \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) & \begin{array}{c} W \\ W^{\perp} \end{array} \\ \begin{array}{cc} W & W^{\perp} \end{array} & \end{array}$$

Note that $P_W^2 = P_W$ (idempotent).
Hence we also have

$$\begin{aligned}(I-P_W)^2 &= I - 2P_W + P_W^2 \\ &= I - 2P_W + P_W \\ &= I - P_W \qquad \text{(also idempotent)}\end{aligned}$$

In fact, $I - P_W = P_{W^\perp}$.

Note that

$$\begin{aligned}P_W P_{W'} &= P_W(I-P_W) \\ &= P_W - P_W^2 = P_W - P_W = 0\end{aligned}$$

and similarly $P_{W'} P_W = 0$.

Thus we say that P_W and $P_{W^\perp}$ are "orthogonal idempotents"

In fact every orthogonal sum arises in this way.

↓

Theorem: We have

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

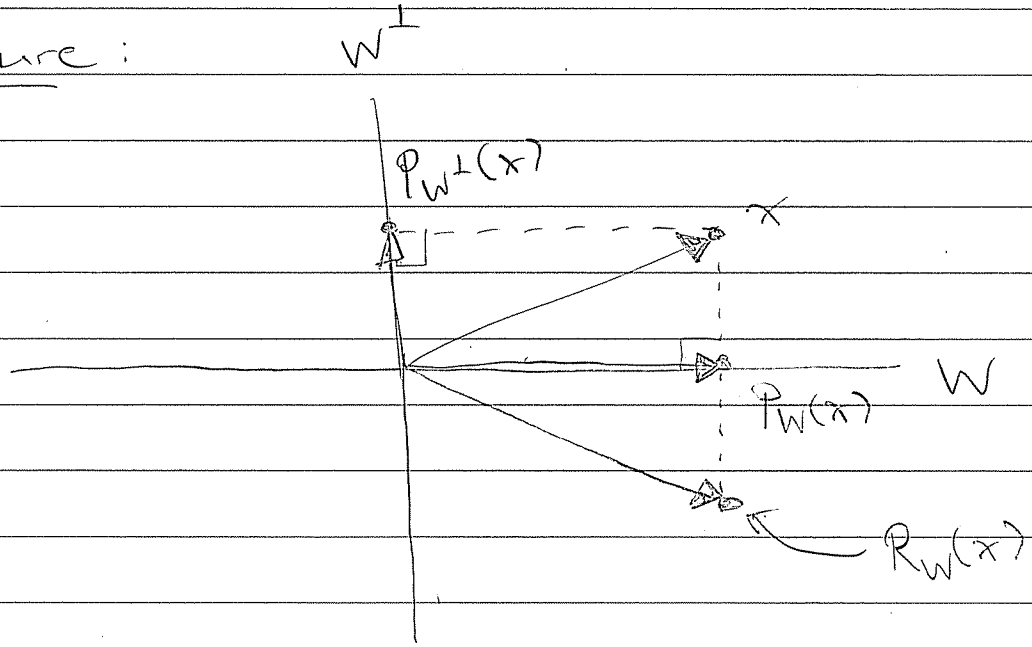
if and only if $\exists$ (unique) elements $e_1, e_2, \ldots, e_k \in O(V,Q)$ such that

① $e_i^2 = e_i$ (idempotents)
② $e_ie_j = e_je_i = 0$ (orthogonal)
③ $I = e_1 + e_2 + \cdots + e_k$ (complete).

Say: "a complete system of orthogonal idempotents"

Proof: Omitted. ▨

Picture:



We can also reflect across a non-degenerate subspace $W \subseteq V$. Define

$$\begin{aligned} S_W(x) &:= x - 2P_{W^\perp}(x) \\ &= (I - 2P_{W^\perp})(x) \\ &= (I - 2(I - P_W))(x) \\ &= (2P_W - I)(x). \end{aligned}$$

$$\boxed{S_W = 2P_W - I}$$

Reflection across W

Q: S_W in coordinates?

Choose bases β for W and $\beta^\perp$ for $W^\perp$ so $\beta \cup \beta^\perp$ is a basis for V.

Let $B(x,y) = \frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right]$

and $B(x,y) = x^t B y$ (matrices)

By definition $\forall x \in V$ we have

$$P_{W^\perp} x \in W^\perp$$

How to say this with matrices?

If $\beta = \{v_1, v_2, \ldots, v_k\}$ then let

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$$

Note that $P_{W^\perp} x \in W^\perp$

$\iff B(v_i, P_{W^\perp} x) = v_i^t B P_{W^\perp} x = 0 \quad \forall v_i \in \beta.$

$\iff A^t B P_{W^\perp} x = 0 \quad ⊛$

Also note that since $P_W x \in W$ we have

$$P_W x = Ay$$

for some β-coordinate vector $y \in W$.

Finally, we use this to "solve" ⊛ :

$$A^t B P_{W^\perp} x = 0.$$
$$A^t B (I - P_W) x = 0.$$
$$A^t B x = A^t B P_W x = A^t B A y.$$

$$\implies y = \underset{\substack{\uparrow \\ \text{invertible since } B \text{ is invertible} \\ \text{and } A \text{ has full rank}}}{(A^t B A)^{-1}} A^t B x$$

Hence

$$P_W x = Ay = A(A^tBA)^{-1}A^tB$$

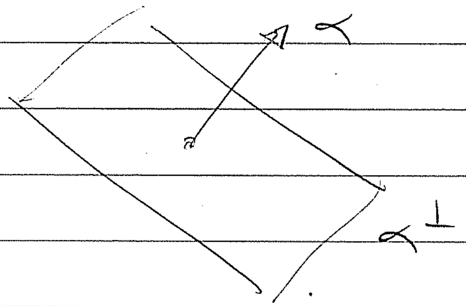
$$\Longrightarrow \boxed{P_W = A(A^tBA)^{-1}A^tB} \in End(V)$$

Check: If $W = V$, i.e. A is square invertible, then

$$\begin{aligned} P_V &= A(A^tBA)^{-1}A^tB \\ &= \cancel{AA^{-1}}B^{-1}\cancel{(A^t)^{-1}A^t}B \\ &= \cancel{B^{-1}B} \\ &= I \end{aligned} \qquad \checkmark$$

Most important case:
Projection onto a hyperplane $\alpha^\perp$

Let $Q(\alpha) \neq 0$, so $V = \underset{\substack{\nearrow \\ \text{line}}}{\alpha} \perp \underset{\substack{\uparrow \\ \text{hyperplane}}}{\alpha^\perp}$



Then
$$\begin{aligned} P_{\alpha^\perp} &= I - P_\alpha \\ &= I - \alpha \overset{\text{scalar}}{\boxed{(\alpha^t B \alpha)^{-1}}} \alpha^t B \\ &= I - \frac{\boxed{\alpha\alpha^t B}}{\boxed{\alpha^t B \alpha}} \begin{matrix} \leftarrow \text{matrix} \\ \leftarrow \text{scalar} \end{matrix} \end{aligned}$$

Thus
$$\begin{aligned} P_{\alpha^\perp} x &= x - \alpha \boxed{\frac{\alpha^t B x}{\alpha^t B \alpha}} \leftarrow \text{scalar.} \\ &= x - \frac{B(\alpha, x)}{B(\alpha, \alpha)} \alpha \end{aligned}$$

Reflection across hyperplane $\alpha^\perp$:

$$S_{\alpha^\perp} = 2 P_{\alpha^\perp} - I$$

$$S_{\alpha^\perp} x = 2\left(x - \frac{B(\alpha, x)}{B(\alpha, \alpha)} \alpha \right) - x$$

☆ $$\boxed{S_{\alpha^\perp}(x) = x - 2\frac{B(\alpha, x)}{B(\alpha, \alpha)} \alpha}$$ ☆

Equation of a Reflection

Concrete Example:

Reflect across plane $x - 2y - z = 0$ in $\mathbb{R}^3$ using the dot product $B = I$

Project onto line $\alpha = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$P_\alpha = \frac{\alpha \alpha^t B}{\alpha^t B \alpha} = \frac{\alpha \alpha^t}{\alpha^t \alpha}$$

$$= \frac{\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} (1 \ -2 \ -1)}{(1 \ -2 \ -1) \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}}$$

$$= \frac{1}{6} \begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

Project onto plane $\alpha^\perp$:

$$P_{\alpha^\perp} = I - P_\alpha$$

$$= \frac{1}{6} \begin{pmatrix} 6 & & \\ & 6 & \\ & & 6 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{pmatrix}$$

Reflection matrix across $\alpha^{\perp}$:

$$S_{\alpha^\perp} = 2P_{\alpha^\perp} - I$$

$$= \frac{1}{6}\begin{pmatrix} 10 & 4 & 2 \\ 4 & 4 & -4 \\ 2 & -4 & 10 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 6 & & \\ & 6 & \\ & & 6 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 4 & 4 & 2 \\ 4 & -2 & -4 \\ 2 & 4 & 4 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix}$$

"Householder matrix"

Check that it works :

Reflect $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ across $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^{\perp}$

$$S_{\alpha^\perp}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{3}\left(1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} - 1\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right)$$

$$\downarrow$$

$$= \frac{1}{3}\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$ ☺

It works!