Thur Oct 23

New Topic.

Q: What is "geometry"?

A: Klein's Erlangen Program (1872).

Suppose X is a "geometry" (set with geometric structure, whatever that means).

Suppose group G acts on X, preserving geometry, i.e. $G \leq \mathrm{Aut}(X)$

Given $x \in X$ define

"orbit" $G(x) = \{ g(x) : g \in G \} \subseteq X$

"stabilizer" $G_x = \{ g : g(x) = x \} \leq G$.

Theorem (Orbit-Stabilizer): $\exists$ Canonical bijection preserving G-action

$$G(x) \approx_G G/G_x \quad \text{"coset space"}$$

$$g(x) \in X \longleftrightarrow gG_x \in G/G_x$$

If the action $G \curvearrowright X$ is transitive (i.e. $\forall x, y \in X \; \exists g \in G$ with $g(x) = y$) then there is only one orbit.

Hence for any $x \in X$ we have

$$X \approx_G G/G_x .$$

☆ Klein's revolutionary idea:

Replace the "geometry" X by the "coset space" G/G_x.
(also called a "homogeneous space").

eg. let $\mathbb{R}P^n$ be the "space" of lines through the origin in $\mathbb{R}^{n+1}$. The group $SO(n+1)$ acts transitively on these lines.
The stabilizer of any particular line is $SO(n+1)_{line} \approx O(n)$. Hence

$$\mathbb{R}P^n \approx SO(n+1)/O(n).$$

This is not just notation. It actually tells us how to "topologize"/"geometrize" projective space.

Thurs Oct 25.

Recall Klein's "Erlangen Program" (1872):

If G acts transitively on X (preserving "geometric" structure) then for any "point" $x \in X$ we have

$$X \approx_G \underbrace{G/G_x}_{\substack{\text{a "space" of cosets} \\ \text{(a "homogeneous space")}}}$$

Structure can be transferred both ways $X \longleftrightarrow G$.

X can be a topological space, a manifold, or more ...
For us, X is a vector space $(V, \mathbb{F})$ with bilinear form $B : V \times V \to \mathbb{F}$.

DEF: The "orthogonal group" relative to B:

$$O(V,B) = \{ \varphi \in GL(V) : B(x,y) = B(\varphi(x), \varphi(y)) \ \forall x, y \in V \}.$$

Could also say $O(V,B) = \mathrm{Aut}(V,B)$.

If we think of (V, B) as a "metric" structure then $O(V, B)$ is the group of "isometries" of (V, B).

We write $B(x, y) = B(\varphi(x), \varphi(y))$ in coordinates as

$$
\begin{aligned}
[x]^t [B][y] &= ([\varphi][x])^t [B] ([\varphi][y]) \\
&= [x]^t ([\varphi]^t [B][\varphi]) [y].
\end{aligned}
$$

Since this holds $\forall\, x, y \in V$ we have

$$[\varphi]^t [B][\varphi] = [B]$$

as matrices.

Thus given any matrix $A \in \mathrm{Mat}_n(\mathbb{F})$, let

$$O(\mathbb{F}^n, A) := \{ P \in GL_n(R) : P^t A P = T \}.$$

Verify it's a group:

(1) $I^t A I = T$.

(2) If $P \in O(\mathbb{F}^n, A)$ then

$$
\begin{aligned}
P^t A P = A \implies A &= (P^t)^{-1} A P^{-1} \\
&= (P^{-1})^t A (P^{-1}).
\end{aligned}
$$

③ If $P, Q \in O(\mathbb{F}^n, A)$ then

$$(PQ)^t A (PQ) = Q^t (P^t A P) Q$$
$$= Q^t A Q$$
$$= A.$$

Trivial case: If $A = 0$ we have

$$O(\mathbb{F}^n, A) = GL(\mathbb{F}^n).$$ "Her All-Embracing Majesty" (Weyl).

Now suppose that $\det A \neq 0$ (i.e the form is non-degenerate. Then we have

$$P^t A P = A$$
$$\Rightarrow \det(P)^2 \cancel{\det(A)} = \cancel{\det(A)}$$
$$\Rightarrow \det(P)^2 = 1$$
$$\Rightarrow \det(P) = \pm 1.$$

We get a decomposition

$$O(\mathbb{F}^n, A) = \underbrace{O^+(\mathbb{F}^n, A)}_{\det = +1} \cup \underbrace{O^-(\mathbb{F}^n, A)}_{\det = -1}$$

orientation-<u>preserving</u> isometries   orientation-<u>reversing</u> isometries

Note: $O^+(\mathbb{F}^n, A)$ is a group since

$$\det(PQ) = \det(P)\det(Q)$$
$$= (+1)(+1) = +1.$$

We say $SO(\mathbb{F}^n, A) := O^+(\mathbb{F}^n, A)$
"special orthogonal" group

Warning: $O^-(\mathbb{F}^n, A)$ is not a group.

Important Special Cases:

Let $(V, B) = (\mathbb{R}^n, \overset{I}{\text{dot product}})$. Then

$$O(\mathbb{R}^n, I) = \{A \in GL(\mathbb{R}^n) : A^t A = I\}.$$

$A^t A = I$ means the columns (resp. rows) of A form an orthonormal basis for $\mathbb{R}^n$.

Say $O(n) := O(\mathbb{R}^n, I)$
"the" orthogonal group.

In genereral consider a symmetric, real, non-deg form

$$I_{p,m} = \left(\begin{array}{c|c} I_p & \\ \hline & -I_m \end{array}\right)$$

DEF: $O(p,m) := O(\mathbb{R}^n, I_{p,m})$.

eg $O(3,1)$ = the Lorentz group of special relativity

$$(x,y,z,t)\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = x^2+y^2+z^2-t^2$$

Isotropic vectors form the "light cone".

Now let $B: V\times V \to \mathbb{F}$ be non-deg and antisymmetric: $B(x,y) = -B(y,x)$ (called a "symplectic form")

It's true (but we didn't prove) that B is equivalent to

$$J_{2n} = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

We define the "symplectic group"

$$Sp(\mathbb{F}^n) := O(\mathbb{F}^n, J_{2n})$$

Finally, what if the form B is general (neither symmetric nor antisymmetric)?

Then ($\operatorname{char} \mathbb{F} \neq 2$) we can "polarize"

$$B(x,y) = B^s(x,y) + B^a(x,y), \text{ where}$$

$$B^s(x,y) = \frac{B(x,y) + B(y,x)}{2} \text{ is symmetric}$$

$$B^a(x,y) = \frac{B(x,y) - B(y,x)}{2} \text{ is antisymmetric.}$$

The polarization is unique.

Now consider a degree 2 field extension

$$\mathbb{F} \subseteq \mathbb{F}[\alpha] = \{a + b\alpha : a, b \in \mathbb{F}\}$$

(assume $\alpha^2 \in \mathbb{F}$).

Then we can "lift" B up to $\mathbb{F}[\alpha]$ and define $h_B : V \times V \to \mathbb{F}[\alpha]$ by

$$h_B(x,y) := B^s(x,y) + \alpha B^a(x,y)$$

Observe that

$$h_B(y,x) = B^s(y,x) + \alpha B^a(y,x)$$
$$= B^s(x,y) - \alpha B^a(x,y) = \overline{h_B(x,y)}$$

where $a+\alpha b \mapsto a-\alpha b = \overline{a+\alpha b}$
is the unique nontrivial Galois automorphism
$\in Gal(\mathbb{F}[\alpha]/\mathbb{F})$. Call it "conjugation".

The form is called "hermitian", or
"sesquilinear" (1 and 1/2 linear)

The group $O(\mathbb{F}[\alpha], h_B)$ is called "unitary"

eg. Luckily $\mathbb{R}$ has a degree 2 extension
called $\mathbb{C}$. Define the standard
hermitian form $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by.

$$\langle x,y \rangle := x^t \bar{y}$$
$$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$$

In general let

$$\langle x,y \rangle_A = x^t A \bar{y}$$

Then we define

$$U(n) := O(\mathbb{C}^n, \langle \circ, \circ \rangle_I)$$
$$U(p,m) := O(\mathbb{C}^n, \langle \circ, \circ \rangle_{I_{p,m}})$$

Good News: You've almost seen it all.

Theorem (Frobenius 1877, Hurwitz 1898):
Classification of <u>real</u> <u>division</u> <u>algebras</u>.

The only commutative ones are

$$\mathbb{R} \underset{2}{\subseteq} \mathbb{C}$$

The only associative ones are

$$\mathbb{R} \underset{2}{\subseteq} \mathbb{C} \underset{2}{\subseteq} \mathbb{H}$$

quaternions

The only "normed" ones are

$$\mathbb{R} \underset{2}{\subseteq} \mathbb{C} \underset{2}{\subseteq} \mathbb{H} \underset{2}{\subseteq} \mathbb{O}$$

octonions.

☆ Big Theorem (Lie, Cartan-Killing, Weyl):

Almost every Lie group looks like $O((V,\mathbb{F}),B)$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and B is hermitian

[Weyl called these the "classical groups"
GL, SL, O, SO, U, SU, Sp]

There are a few exceptions related to $\mathbb{O}$.

☆☆ Huge Theorem (Dickson, Chevalley, etc.)

Almost every finite simple group looks like a Lie group over a finite field.

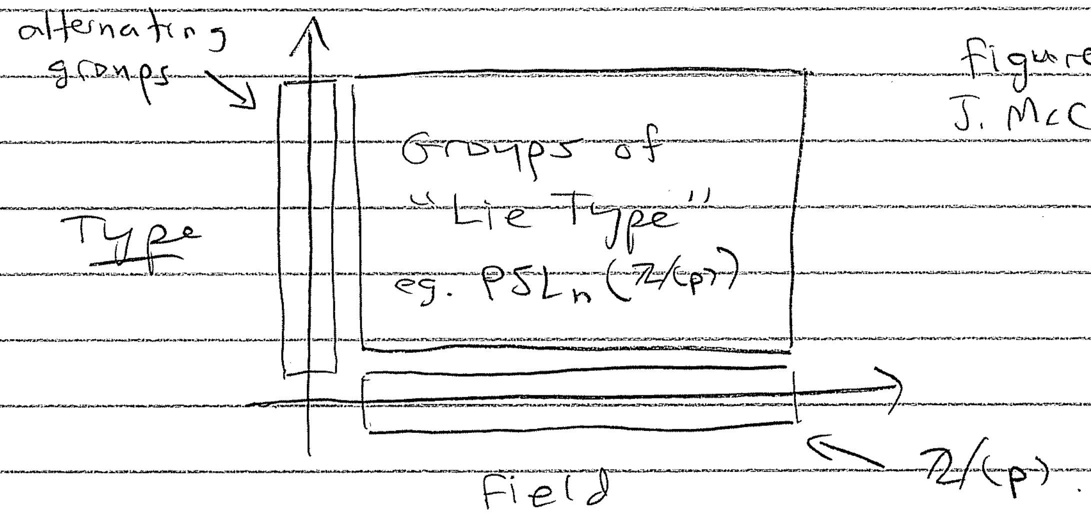

figure from J. McCammond.

+ 26 "sporadic" exceptions.