Tues Oct 9

What is "geometry"?

DEF: Let $(V, \mathbb{F})$ be a f.d. vector space. We say $B : V \times V \to \mathbb{F}$ is a "bilinear form" if $\forall x \in V$ the functions

$$\begin{array}{ccc} V \to \mathbb{F} & \& & V \to \mathbb{F} \\ y \mapsto B(x,y) & & y \mapsto B(y,x) \end{array}$$

are $\mathbb{F}$-linear.

Example: Let $V = \mathbb{F}^n$ and let $A \in \mathrm{Mat}_n(\mathbb{F})$ be any $n \times n$ matrix. Then

$$B(x,y) := x^t A y$$

is a bilinear form (Exercise). Explicitly, we have

$$B(x,y) = (x_1, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= (x_1, \ldots, x_n) \begin{pmatrix} \sum_j y_j a_{1j} \\ \vdots \\ \sum_j y_j a_{nj} \end{pmatrix}$$

$$= \sum_i x_i \left(\sum_j y_j a_{ij} \right)$$

$$= \sum_{i,j} x_i y_j a_{ij}$$

$$= x^i y^j a_{ij} \text{ in "Einstein notation"}$$

If $A = I$ this is called the standard bilinear form

$$B(x,y) = x^t I y$$
$$= x^t y$$
$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

"dot product"

[Remark: This works best for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, but even over general $\mathbb{F}$ we can still do "geometric" things with a bilinear form

e.g "lengths"
"angles"
"reflections"]

Conversely, let $(V, \mathbb{F})$ be an <u>abstract</u> vector space of $\dim = n$ and choose a basis $\beta = \{v_1, v_2, \ldots, v_n\} \subseteq V$.

Then any bilinear $B : V \times V \to \mathbb{F}$ can be written in coordinates.

<u>DEF</u>: Let

$[B]_\beta$ = $n \times n$ matrix with i,j entry $B(v_i, v_j)$

(the "Gram matrix" of β with respect to B)

Then $\forall\, x, y \in V$ we have

$$B(x,y) = [x]_\beta^t \, [B]_\beta \, [y]_\beta$$

Proof: Let $x = \sum x_i v_i$, $y = \sum y_j v_j$

Then $B(x,y) = B(\sum x_i v_i, \sum y_j v_j)$

$$= \sum_{i,j} x_i y_j B(v_i, v_j)$$

$$= (x_1, \ldots, x_n) \begin{pmatrix} B(v_1,v_1) & B(v_1,v_2) & & \\ B(v_2,v_1) & B(v_2,v_2) & & \\ & & \ddots & \\ & & & B(v_n,v_n) \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= [x]_\beta^t [B]_\beta [y]_\beta \quad \blacksquare$$

So...

Q: what is a "matrix"?

Answer 3: A bilinear form written in coordinates

[Just like Answers 1 & 2, this leads to a point of view.]

Change of Coordinates:

Let $\alpha = \{u_1, \ldots, u_n\}$ and $\beta = \{v_1, \ldots, v_n\}$ be two bases for $(V, \mathbb{F})$.

Recall, $\forall x \in V$ we have

$$[x]_\beta = C_{\alpha\beta} [x]_\alpha$$

where $C_{\alpha\beta} = \left[[u_1]_\beta \; [u_2]_\beta \; \cdots \; [u_n]_\beta \right]$

and $C_{\beta\alpha} = C_{\alpha\beta}^{-1} = \left[[v_1]_\alpha \; [v_2]_\alpha \; \cdots \; [v_n]_\alpha \right]$

"change of basis matrix".

Now consider bilinear $B : V \times V \to \mathbb{F}$.

How are $[B]_\alpha$ and $[B]_\beta$ related?

$\forall \, x, y \in V$ we have

$$\begin{aligned} [x]_\alpha^t [B]_\alpha [y]_\alpha &= B(x,y) \\ &= [x]_\beta^t [B]_\beta [y]_\beta \end{aligned}$$

$$= (C_{\alpha\beta}[x]_\alpha)^t [B]_\beta (C_{\alpha\beta}[y]_\alpha)$$

$$= [x]_\alpha^t \left(C_{\alpha\beta}^t [B]_\beta C_{\alpha\beta} \right) [y]_\alpha .$$

Letting x, y range over u_i, u_j gives

$$\boxed{[B]_\alpha = C_{\alpha\beta}^t [B]_\beta C_{\alpha\beta}}$$

Is this conjgacy? NO

Summary: Matrices $A, A' \in \mathrm{Mat}_n(\mathbb{F})$ represent the same bilinear form in different coordinates

$$\Updownarrow$$

$\exists$ invertible $P \in \mathrm{Mat}_n(\mathbb{F})$ such that

$$A' = P^t A P.$$

WARNING: This is NOT conjugation.
(unless $P^t = P^{-1}$.)

The theories of

linear operators $\neq$ bilinear forms
(Answer 1) (Answer 3)

Note

$$\det(A') = \det(P^t)\det(A)\det(P)$$
$$= \det(A)\det(P)^2$$
$$\neq \det(A)$$
in general.

"Determinant is not a property of bilinear forms"

However, since $\det(P) \neq 0$ we do have

$$\det(A') = 0 \iff \det(A) = 0.$$

Maybe this <u>means</u> something

"Duality"

Given a vector space $(V, \mathbb{F})$ let

$$V^* = \mathrm{Hom}(V, \mathbb{F})$$
$$= \{\text{linear functions } V \to \mathbb{F}\}.$$

As we know, V^* is itself a <u>vector space</u> over $\mathbb{F}$: Given $f, g \in \mathrm{Hom}(V, \mathbb{F})$ and $\alpha \in \mathbb{F}$ we define $\alpha f + g \in \mathrm{Hom}(V, \mathbb{F})$

by $(\alpha f + g)(x) := \alpha(f(x)) + g(x)$

$\forall x \in V$. Now suppose $\dim(V) = n < \infty$ with basis $\beta = \{v_1, \ldots, v_n\} \subseteq V$.

Exercise: Then $\dim(V^*) = n$ with "dual basis" $\beta^* = \{v_1^*, \ldots, v_n^*\} \subseteq V^*$

defined by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

coordinate functions.

Note: $v_i^*(x) = v_i^*(x_1 v_1 + \cdots + x_n v_n)$
$= x_1 v_i^*(v_1) + \cdots + x_n v_i^*(v_n)$
$= x_i$
the ith coordinate of x.

Now every bilinear $B : V \times V \to \mathbb{F}$ determines a morphism

$$V \xrightarrow{\varphi_B} V^*$$
$$x \longmapsto B(x, \cdot)$$

by sending x to the linear function $y \longmapsto B(x,y)$.

DEF: We say B is "non-degenerate" $\iff$ φ_B is an isomorphism $V \approx V^*$

(also called a "pairing" between V and V^*).

If $\dim(V) < \infty$, we can represent φ_B as a matrix.

Claim: Choose basis $\beta = \{v_1, \cdots, v_n\} \subseteq V$. Then

$$[\varphi_B]_{\beta\beta^*} = [B]_\beta^t = (B(v_j, v_i))_{i,j}$$

<u>Proof</u>. By definition we have $\forall x \in V$,

$$[\varphi_B]_{\beta\beta^*} [x]_\beta = [B(x, \cdot)]_{\beta^*}$$

So the ith column of $[\varphi_B]_\beta$ is

$$[\varphi_B]_{\beta\beta^*} [v_i]_\beta = [B(v_i, \cdot)]_{\beta^*}$$

Express $B(v_i, \cdot) : V \to \mathbb{F}$ in terms of the basis $v_1^*, v_2^*, \dots, v_n^* : V \to \mathbb{F}$.

$\forall x = x_1 v_1 + \cdots + x_n v_n \in V$ we have

$$\begin{aligned} B(v_i, x) &= B(v_i, x_1 v_1 + \cdots + x_n v_n) \\ &= B(v_i, v_1) x_1 + \cdots + B(v_i, v_n) x_n \\ &= B(v_i, v_1) v_1^*(x) + \cdots + B(v_i, v_n) v_n^*(x) \end{aligned}$$

$$= \left(B(v_i, v_1) v_1^* + \cdots + B(v_i, v_n) v_n^* \right)(x).$$

coefts of $B(v_i, \cdot)$ w.r.t. β^*

Conclusion: The i,j-entry of $[\varphi_B]_{\beta\beta^*}$ is $B(v_i, v_j)$.

$$\Longrightarrow [\varphi_B]_{\beta\beta^*} = [B]_\beta^t \qquad \blacksquare$$

Corollary: Form $B: V \times V \to \mathbb{F}$ is non-degen.

$$\Updownarrow$$

$$\det \varphi_B \neq 0.$$

$$\Updownarrow$$

$$\det B \neq 0.$$

[Mantra: A non-degenerate 2-form $V \times V \to \mathbb{F}$ defines a non-canonical bijection between V and V^*.]