Thurs Oct 4

Take a breath ...

Where were we?

- What is "space"?
- What is a "matrix"?

(1) An endomorphism in coordinates

(2) A weighted, directed graph.

2 leads to linear dynamics

- powers of matrices
- Perron-Frobenius (positivity)
- spectrum of a graph
- Coxeter graphs of type A, D, E.
- A mysterious game.

Where next?

- What is "geometry".
- What is a "matrix"

(3) A <u>bilinear</u> <u>form</u> in coordinates

3 leads to geometry

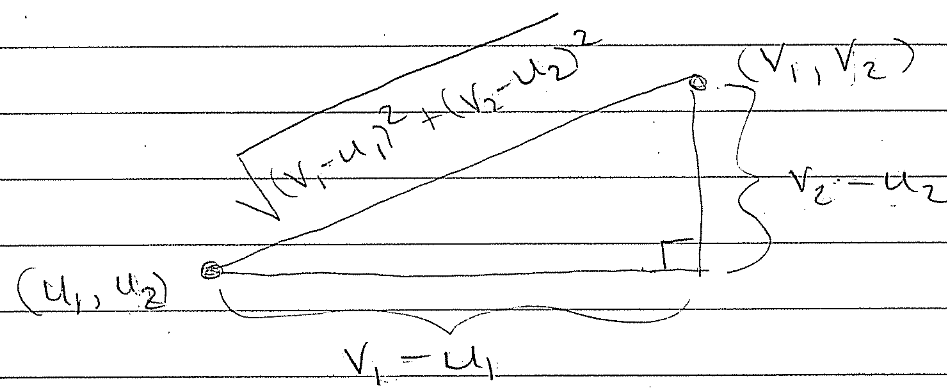
- "positivity" of forms
- projection / reflection / rotation
- Cartan-Dieudonné
- groups generated by reflections

So let's begin.

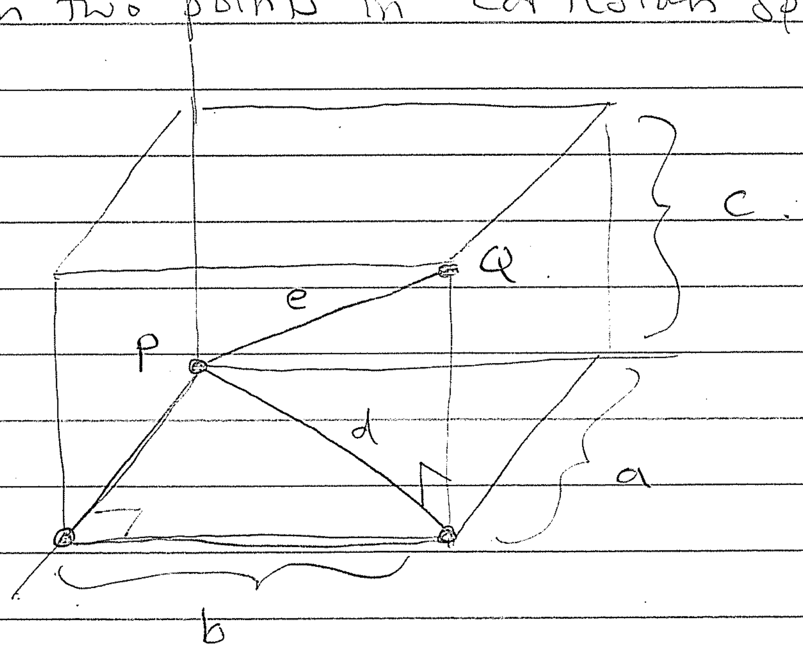
Q: What is "geometry"?

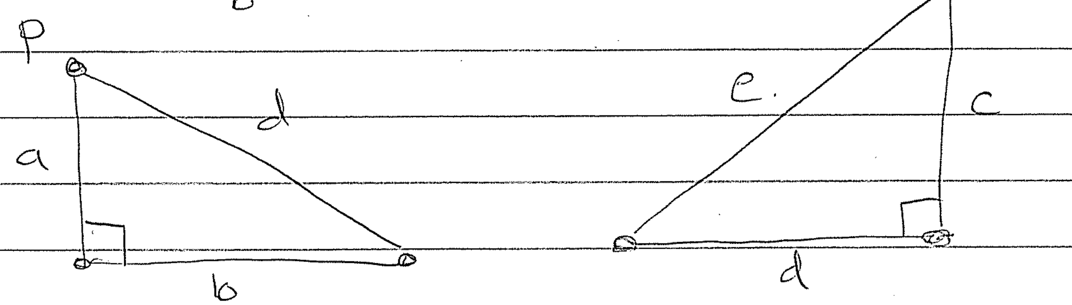
A: The Pythagorean Theorem

Given two points in the "Cartesian plane"



Given two points in "Cartesian space"





Pythagorus : $a^2 + b^2 = d^2$

$$d^2 + c^2 = e^2$$

$$\Longrightarrow \quad e^2 = a^2 + b^2 + c^2$$

$$e = \sqrt{a^2 + b^2 + c^2}$$

$$\operatorname{dist}\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right)$$

$$= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$$

Hamilton (1843) invented quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Think of a "purely imaginary" quaternion as a point in $\mathbb{R}^3$

$$u = u_1 i + u_2 j + u_3 k$$

$$v = v_1 i + v_2 j + v_3 k.$$

Quaternion multiplication gives

$$uv = -(u_1 v_1 + u_2 v_2 + u_3 v_3)\, 1$$

$$+ \left[(u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k \right]$$

i.e. $uv = -u \cdot v + u \times v$.

Real part $\in \mathbb{R}$ "imaginary part" $\in \mathbb{R}^3$.

This gave Gibbs (~1881) and Heaviside (~1883) a good idea: get rid of i, j, k.

Define "vectors" $\vec{x} = (x_1, x_2, x_3)$ with 2 operations

Dot Product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + u_3 v_3$$

Cross Product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{u} \times \vec{v} := (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

[Also hidden in Grassmann (1844)]

Tait didn't like it. (~1890)
- a "hermaphrodite monster"

[Compare Leibniz:
"an amphibian between being and non-being."]

Things settled down and we got the notion of an inner-product space

DEF: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An inner product space is a vector space $(V, \mathbb{F})$ together with a map $\langle , \rangle : V \times V \to \mathbb{F}$ (called a "Hermitian inner product") satisfying $\forall\, x, y, z \in V$ and $\alpha \in \mathbb{F}$

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0 \iff x = 0$.

DISCUSS

From this we can define a "metric structure": $\forall\, x \in V$, let

$$\|x\|^2 := \langle x, x \rangle$$

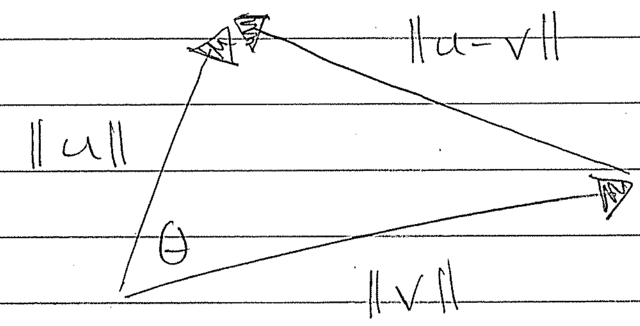
(The Pythagorean "Theorem" has become a "definition")

The "distance" between points is

$$d(x, y)^2 := \|x - y\|^2$$

$$= \langle x - y, x - y \rangle$$

Since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ we can draw a picture:



On one hand:

$$\begin{aligned}\|u-v\|^2 &= \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle - 2\langle u,v\rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle u,v\rangle.\end{aligned}$$

On the other: (Law of Cosines)

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$$

This suggests a further <u>definition</u>

$$\text{angle}(x,y) := \arccos\left(\frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}\right)$$

Moral: An "inner product" allows us to <u>define</u> $\underbrace{\text{lengths and angles}}_{\text{"geometry"}}$ and "orthogonality"

Remark: The axiom

- $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0 \iff x = 0$

says that $\langle , \rangle$ is "positive-definite".
For this we need an ordered field like $\mathbb{R}$.

But much of theory of "geometric algebra" is more general.

DEF: Let $(V, \mathbb{F})$ be any vector space. We say that $B: V \times V \to \mathbb{F}$ is a bilinear form if $\forall x \in V$ the functions

$$\begin{aligned} V &\longrightarrow \mathbb{F} \\ y &\longmapsto B(x,y) \end{aligned} \qquad \& \qquad \begin{aligned} V &\longrightarrow \mathbb{F} \\ y &\longmapsto B(y,x) \end{aligned}$$

are $\mathbb{F}$-linear.

Example: Let $V = \mathbb{F}^n$ and consider a matrix $A \in M_n(\mathbb{F})$.
Then the function

$$B(x,y) := x^t A y$$

is bilinear.

i.e. $B(x,y) = (x_1, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$= (x_1, \ldots, x_n) \begin{pmatrix} \sum_j y_j a_{1j} \\ \vdots \\ \sum_j y_j a_{nj} \end{pmatrix}$$

$$= \sum_i \left(x_i \sum y_j a_{ij} \right)$$

$$= \sum_{i,j} x_i y_j a_{ij}$$

($= x^i y^j a_{ij}$ in "Einstein notation")

If $A = I$, this is called the <u>standard</u> bilinear form

$$B(x,y) = x^t I y$$
$$= x^t y$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Conversely, let $(V, \mathbb{F})$ be an abstract vector space of $\dim = n$ and choose a basis $\beta = \{v_1, v_2, \ldots, v_n\} \subseteq V$.

Then a bilinear form $B: V \times V \to \mathbb{F}$ can be written in coordinates.

Define the matrix

$$[B]_\beta := i,j\text{-entry } B(v_i, v_j)$$

Then: $\forall\, x, y \in V$ we have

$$B(x,y) = [x]_\beta^t [B]_\beta [y]_\beta$$

Proof: Let $x = \sum x_i v_i$, $y = \sum y_j v_j$. Then

$$B(x,y) = B\left(\sum x_i v_i, \sum y_j v_j\right)$$

$$= \sum_{i,j} x_i y_j B(v_i, v_j)$$

$$= [x]_\beta^t [B]_\beta [y]_\beta$$

$\blacksquare$