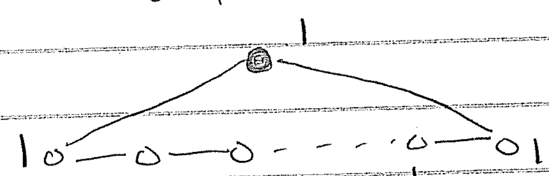
Thurs Sept 27

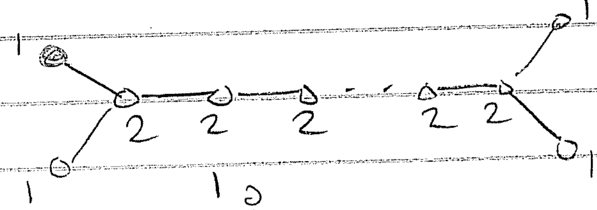
<u>Very Big Day</u>

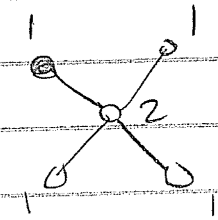
Lemma: If $A \geq 0$ is connected and $Av = \lambda v$ with $v \gneq 0$, then $\rho(A) = \lambda$.

<u>Proof</u>: By PF $\exists\, w^T > 0$ with $w^T A = \rho(A) w^T$.
Then $\rho(A) w^T v = w^T A v = \lambda w^T v$.
Since $w^T > 0$ and $v \gneq 0 \Rightarrow w^T v \neq 0$
we cancel to get $\rho(A) = \lambda$ $\boxed{}$

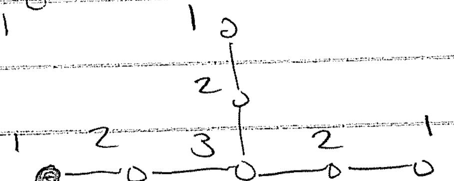
Oddly Specific Lemma:
The following graphs G have $\rho(G) = 2$,

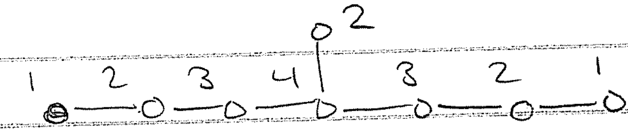
$A_\ell^{(1)} =$


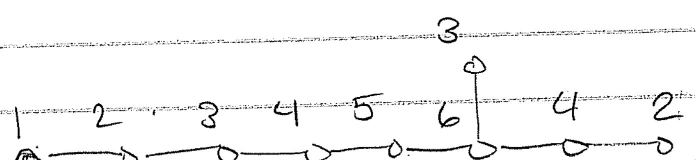
$D_\ell^{(1)} =$


$\left(D_4^{(1)} = \right.$


$\left. \right)$

$E_6^{(1)} =$


$E_7^{(1)} =$


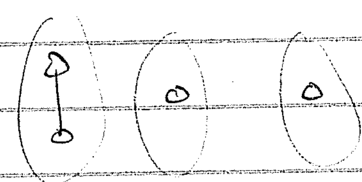
$E_8^{(1)} =$


[Convention: X_ℓ is a graph with ℓ vertices.
$X_\ell^{(1)}$ is a graph with $\ell+1$ vertices.]

Proof of O.S.L.:
I just showed you the PF eivectors v.
Check that $Av = 2v$ in each case.
Apply previous Lemma. $\blacksquare$

well, that was oddly specific (for now)

Now consider a disjoint union of connected graphs $H = H_1 \cup H_2 \cup \cdots \cup H_k$

eg $H =$ 

$H_1 \cup H_2 \cup H_3$.

Then the adjacency matrix is "block-diagonal"

$$A_H = \begin{pmatrix} A_{H_1} & & & \\ & A_{H_2} & & \\ & & \ddots & \\ & & & A_{H_k} \end{pmatrix}$$

Note : $\chi(A_H, \lambda) = \prod_{i=1}^{k} \chi(A_{H_i}, \lambda)$

$\Longrightarrow \{\text{e.values of } H\} = \bigcup_{i=1}^{k} \{\text{e.values of } H_i\}$

$\Longrightarrow \rho(H) = \max_{i=1}^{k} \{\rho(H_i)\} \geq \rho(H_j) \ \forall j.$

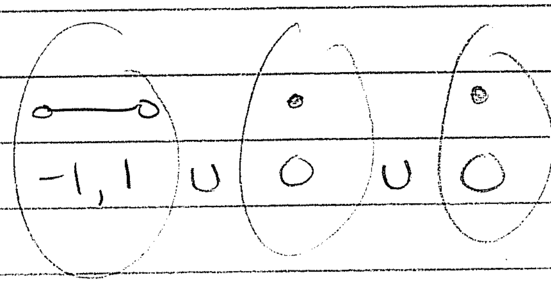
eg $\quad A_H = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$\chi(A_H, \lambda) = (\lambda^2 - 1) \cdot \lambda \cdot \lambda$

$\Longrightarrow$ Spectrum :

H =

-1, 0, 0, 1



Now let G be connected with connected subgraph H

We can write

$$A_G = \begin{pmatrix} A_H & * \\ \hline * & * \end{pmatrix} \geq \overbrace{\begin{pmatrix} A_H & 0 \\ \hline 0 & 0 \end{pmatrix}}^{A_{G|H}} \geq 0$$

[Note: $\rho(A_{G|H}) = \rho(A_H) = \rho(H)$]

Since H is connected, PF $\Rightarrow \exists\, v > 0$ with $A_H v = \rho(H) v$, and hence

$$A_{G|H} \begin{pmatrix} v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_H v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \rho(H) v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} = \rho(H) \begin{pmatrix} v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Define $\tilde{v} = \begin{pmatrix} v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \gneq 0$.

Also, observe that $A_G \tilde{v} \gneq A_{G|H} \tilde{v}$.
(Why? Hint: G is connected.)

Finally, PF $\Rightarrow \exists\, y^T > 0$ with $y^T A_G = \rho(G) y^T$.
So we have

$$\begin{aligned} \rho(G) \cancel{y^T \tilde{v}} &= y^T A_G \tilde{v} \\ &> y^T A_{G|H} \tilde{v} \\ &= \rho(H) \cancel{y^T \tilde{v}} \end{aligned}$$

and

$$y^T \tilde{v} > 0 \Rightarrow \rho(G) > \rho(H).$$

Summary:

☆ Very Big Lemma ☆.

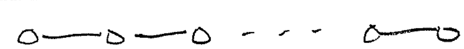
Let G be a connected graph and let H be <u>any</u> proper (ie. $H \neq G$) subgraph.

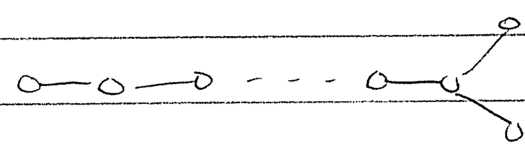
Then $\boxed{\rho(H) < \rho(G)}$

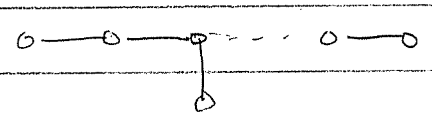
<u>Proof</u>: We did it for connected H. If H is not connected replace H by connected component $H' \subseteq H$ with $\rho(H') = \rho(H)$ ∎

☆ Very Big Theorem ☆

The graphs G with $\rho(G) < 2$ are EXACTLY (disjoint unions of).

A_ℓ =  $(\ell \geq 1)$

D_ℓ =  $(\ell \geq 4)$

E_ℓ =  $(\ell = 6, 7, 8)$

That's All.

Proof: Suppose $\rho(G) < 2$.

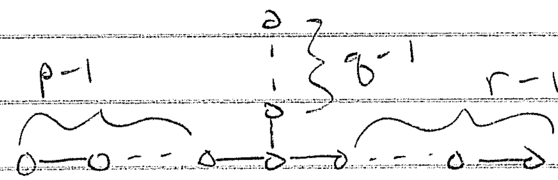
(1) Then $A_\ell^{(1)} \not\subseteq G$ since otherwise
$$2 = \rho(A_\ell^{(1)}) \leq \rho(G)$$

Hence G is a tree (no cycles).

(2) G has no vertex of degree ≥ 4 since otherwise $2 = \rho(D_4^{(1)}) \leq \rho(G)$.

(3) G has at most one vertex of degree 3 since otherwise $2 = \rho(D_\ell^{(1)}) \leq \rho(G)$.

Hence G is a triangle graph

$T_{p,q,r} =$ 

(4) Finally, the obstructions
$E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
tell us that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

∎

Rephrase: The graphs G with $\rho(G) < 2$ are disjoint unions of triangle graphs $T_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

<u>Definition</u>: $A_\ell, D_\ell, E_6, E_7, E_8$ are called "Coxeter Graphs" :)

Thinking Homework: Explain <u>WHY</u>

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

[The only good answer I know involves "McKay correspondence"]
(deep!)

Thinking Homework: Prove that

$$A_\ell^{(1)}, D_\ell^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$$

are the <u>only</u> (simple) graphs with $\rho = 2$.