Tues Sept 25.

Recall: The Spectrum of a Path

$$A = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix} \longleftrightarrow \underbrace{\circ - \circ - \circ \cdots \circ - \circ}_{n \text{ vertices}}$$

call it "A_n"

Goal: Diagonalize A.

Short Answer: Let $k, l \in \mathbb{Z}$. Then we have

$$\sin\left(\frac{lk\pi}{n+1} + \frac{k\pi}{n+1}\right) = \sin\left(\frac{lk\pi}{n+1}\right)\cos\left(\frac{k\pi}{n+1}\right) + \cos\left(\frac{lk\pi}{n+1}\right)\sin\left(\frac{k\pi}{n+1}\right)$$

$$+ \quad \sin\left(\frac{lk\pi}{n+1} - \frac{k\pi}{n+1}\right) = \sin\left(\frac{lk\pi}{n+1}\right)\cos\left(\frac{k\pi}{n+1}\right) - \cos\left(\frac{lk\pi}{n+1}\right)\sin\left(\frac{k\pi}{n+1}\right)$$

$$\sin\left(\frac{(l-1)k\pi}{n+1}\right) + \sin\left(\frac{(l+1)k\pi}{n+1}\right) = \left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\sin\left(\frac{lk\pi}{n+1}\right).$$

So what?

Define $(S)_{lk} = \sin\left(\frac{lk\pi}{n+1}\right)$ $\quad n \times n$ matrix

$$= \left(V_1 \; V_2 \; \cdots \; V_n\right)$$

where $v_k = \begin{pmatrix} \sin\left(\frac{k\pi}{n+1}\right) \\ \sin\left(\frac{2k\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nk\pi}{n+1}\right) \end{pmatrix}$ evector for evalue $2\cos\left(\frac{k\pi}{n+1}\right)$.

Then (*) says that $A v_k = 2\cos\left(\frac{k\pi}{n+1}\right) v_k$.

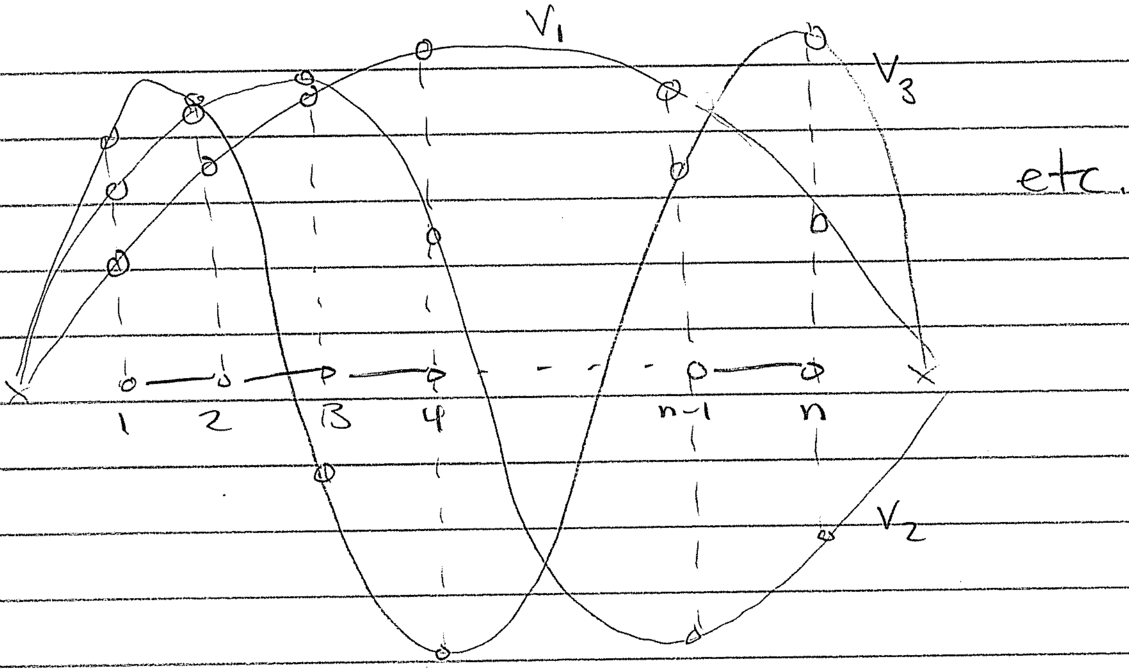
Hence we have

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & 0 \end{pmatrix} = S \begin{pmatrix} 2\cos\left(\frac{\pi}{n+1}\right) & & \\ & \ddots & \\ & & 2\cos\left(\frac{n\pi}{n+1}\right) \end{pmatrix} S^{-1}$$

Computation first done by Lagrange in 1759 (age 23)
"Researches on the Nature and propagation of sound".

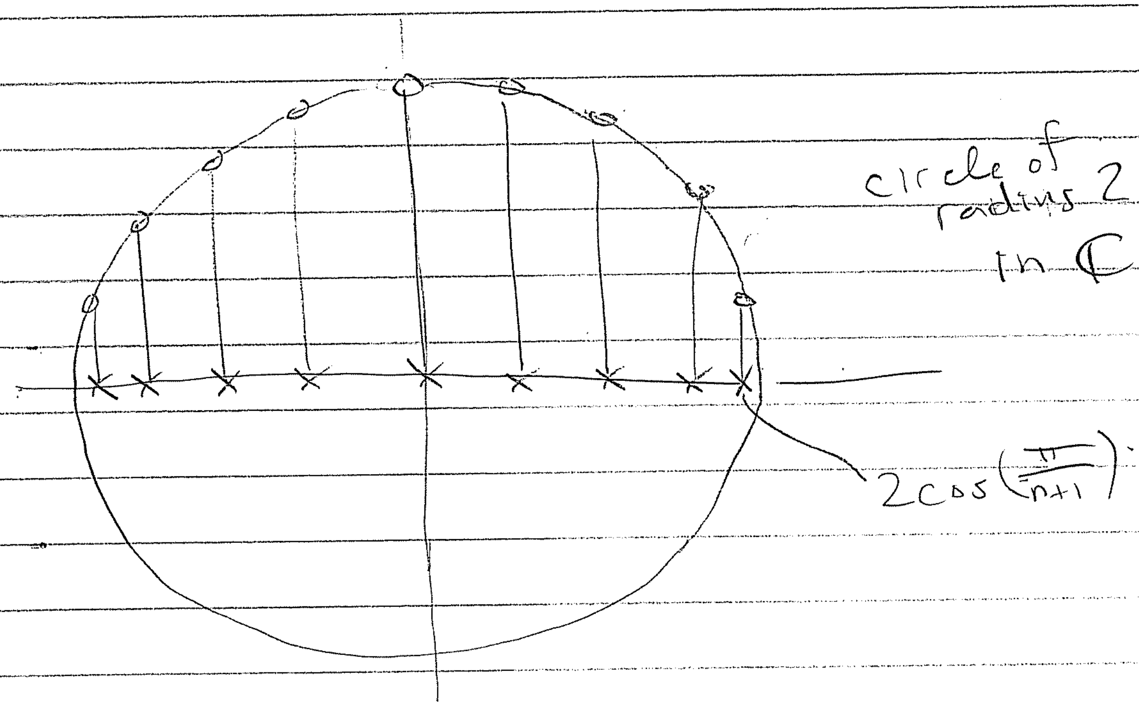
Now we would call it a DFT
(Discrete Fourier Transform).

The eigenvectors are "modes of vibration"



Observe: $\rho(A) = 2\cos\left(\frac{\pi}{n+1}\right) < 2$
with PF eigenvector V_1
(the "fundamental mode")

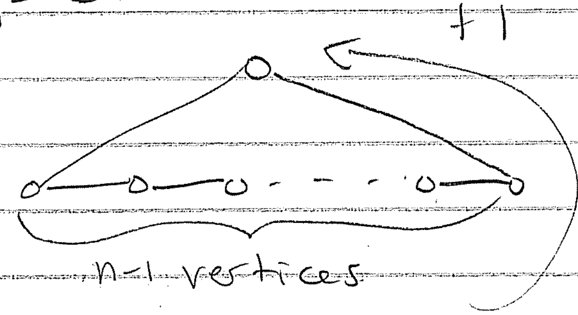
The spectrum of the path is



Q: Which other (simple, undirected) graphs G have $\rho(G) < 2$?

eg. The n-cycle

$A_{n-1}^{(1)} =$



Adjacency matrix $A = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & 0 & \\ & 1 & 0 & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & 1 & 0 \end{pmatrix}$

Easy to see $A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

Claim: This means $\rho(A_{n-1}^{(1)}) = 2$

Proof: Let $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ so $Av = 2v$.

Since A is ≥ 0 and irreducible (the cycle is connected) $\exists\ w^T > 0$ with

$$w^T A = \rho(A)\, w^T .$$

Then we have

$$
\begin{aligned}
\rho(A)(w^T v) &= (\rho(A) w^T) v \\
&= (w^T A) v \\
&= w^T (A v) \\
&= w^T (2v) \\
&= 2 (w^T v).
\end{aligned}
$$

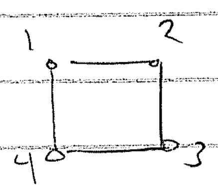
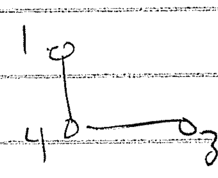
Then $w^T > 0$ and $v > 0 \Rightarrow w^T v \neq 0$

$\Rightarrow \rho(A) = 2$ ∎

In any case, $\rho(A^{(1)}_{n-1}) \not< 2$

☆ Very Important Lemma ☆.

Let G be a connected graph and let H be any subgraph $\neq G$.

e.g. $G =$  , $H =$  .

Then $\boxed{\rho(H) < \rho(G)}$

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<u>Proof</u>: Let G have vertices $V = X \cup Y$, where H has vertices X.

Let A_G = adj. matrix of G.

$A_{G|H}$ = adj. matrix of H with Y rows and columns set to zero.

Then up to conjugacy we have.

$$A_{G|H} = \left(\begin{array}{c|c} A_H & 0 \\ \hline 0 & 0 \end{array}\right)$$

So $\{\text{evalues } A_{G|H}\} = \{0\} \cup \{\text{evalues of } A_H\}$ and hence $\rho(A_{G|H}) = \rho(A_H)$

But by Perron Frobenius, since $0 \leq A_{G|H} \lneq A_G$ with A_G irreducible

we have $\rho(A_{G|H}) < \rho(A_G)$

$\blacksquare$

Corollaries:

① If G has an edge then $\rho(G) \geq 1$.

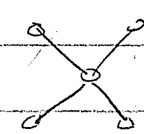
<u>Proof</u>: $1 = \rho(\circ\!\!-\!\!\circ) \leq \rho(G)$ ■

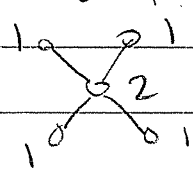
② If G contains a cycle (say an n-cycle $A^{(1)}_{n-1}$) then $\rho(G) \geq 2$.

<u>Proof</u>: $2 = \rho(A^{(1)}_{n-1}) \leq \rho(G)$. ■

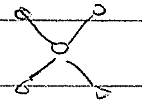
[So which graphs have $1 \leq \rho(G) < 2$?
They are trees (acyclic)
but which trees?]

③ If G has a vertex of degree ≥ 4 then $\rho(G) \geq 2$.

<u>Proof</u>: The graph $D^{(1)}_4 =$  has spectrum 2

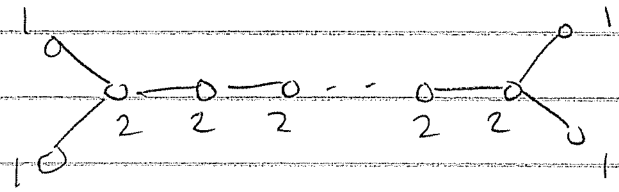
with evector 

. Hence.

$2 = \rho($  $) \leq \rho(G)$.

(4) If G has ≥ 2 vertices of degree 3 then $\rho(G) \geq 2$.

Proof: $D_n^{(1)} =$

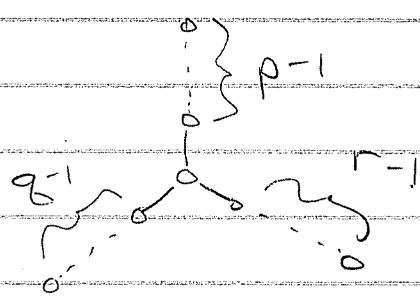


has radius 2 with shown eigenvector. Hence

$$2 = \rho(D_n^{(1)}) \leq \rho(G).$$

What's left? Define the "Triangle Graph"

$T_{p,q,r} =$



Q: for which p, q, r is

$$\rho(T_{p,q,r}) < 2 \quad ??$$