5/30/14

Review of 661/662

Last Time: Abstract Groups
Today: Groups Acting on Things

Q: Why is a group operation associative?
A: To model composition of functions.
In nature, groups occur as automorphisms of things.

$$\mathrm{Aut}(X) := \{ \text{bijections} : X \to X \text{ preserving the structure of } X \}.$$

Examples:

$\mathrm{Aut}(\text{set } X) = \mathrm{Perm}(X)$.

$\mathrm{Aut}(\text{vector space } K^n) = GL_n(K)$

$\mathrm{Aut}(\text{Hermitian space } \mathbb{C}^n) = U(n)$

Let G be an abstract group and consider a group homomorphism

$$\varphi : G \longrightarrow \mathrm{Aut}(X)$$

where X is a nice thing. The pair (X, φ) is called a <u>representation</u> of G.

661 Philosophy (Felix Klein, 1872):
Given a representation $\varphi: G \to \mathrm{Aut}(X)$ we obtain a correspondence

info about G $\longleftrightarrow$ info about X

Simplest Case: If X is just a set ($\mathrm{Aut}(X) = \mathrm{Perm}(X)$) then the pair (φ, X) is called a "G-set".

Equivalently, we say a group G <u>acts</u> on a set X if we have a map

$$G \times X \to X$$
$$(g, x) \mapsto g * x$$

satisfying

- $\forall x \in X$, $1 * x = x$
- $\forall g, h \in G, x \in X$, $(gh)* = g*(h*x)$.

[Equivalence: $g * x = \varphi_g(x) = $ "$g(x)$"]

Given G-sets (X, φ), (Y, ψ) we say $f: X \to Y$ is a G-set morphism if

- $\forall g \in G, x \in X$, $f(\varphi_g(x)) = \psi_g(f(x))$.
  " $f(g(x)) = g(f(x))$ "

☆ The Fundamental Theorem of G-sets:

Every G-set is a disjoint union of transitive G-sets (orbits) and every trans. G-set is isomorphic to G/H for some $H \leq G$. (G acts on G/H by left multiplication.) Furthermore,

$$G/H \approx_G G/K \iff K = gHg^{-1} \text{ for some } g \in G.$$

Special Case (Orbit-Stabilizer):

Let $G \curvearrowright X$. Then for all $x \in X$ we have a <u>bijection</u>

$$G/\mathrm{Stab}(x) \longleftrightarrow \mathrm{Orb}(x).$$
$$g \cdot \mathrm{Stab}(x) \longleftrightarrow g(x).$$

Proof:

$$g \cdot \mathrm{Stab}(x) = h \cdot \mathrm{Stab}(x) \iff g^{-1}h \in \mathrm{Stab}(x)$$
$$\iff g^{-1}h(x) = x$$
$$\iff h(x) = g(x).$$

Application: Let $Gr_K(r,n)$ be the set of r-dim subspaces of K^n. Then $GL_n(K)$ acts transitively on $Gr_K(r,n)$ with stabilizer

$$P_r = \left\{ \begin{array}{c|cc} & \overbrace{\quad}^{r} & \overbrace{\quad}^{n-r} \\ r\{ & A & C \\ n-r\{ & 0 & B \end{array} \right\} \leq GL_n(K).$$

Hence $Gr_K(r,n) \longleftrightarrow GL_n(K)/P_r$

If $K = \mathbb{F}_q$ we know

$$|GL_n(\mathbb{F}_q)| = q^{\binom{n}{2}}(q-1)^n [n]_q!$$

and hence

$$|Gr_{\mathbb{F}_q}(r,n)| = \frac{[n]_q!}{[r]_q!\,[n-r]_q!} = \begin{bmatrix} n \\ r \end{bmatrix}_q$$

Another Application: Let $H, K \leq G$ (possibly both non-normal) and let H act on G/K by left multiplication.

Note that

$$HK = \bigsqcup_{C \in \mathrm{Orb}(K)} C$$

$$\Longrightarrow \quad |HK| = |\mathrm{Orb}(K)| \cdot |K|.$$

We also know $\mathrm{Stab}(K) = H \cap K$ and hence

$$|HK| = \left(\frac{|H|}{|\mathrm{Stab}(K)|} \right) \cdot |K|$$

$$= \frac{|H|}{|H \cap K|} \cdot |K|.$$

$$\Longrightarrow \quad |HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

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Corollary: Given any $x \in G$ define the double coset

$$HxK := \{ hxk : h \in H, k \in K \}.$$

Then we have

$$|HxK| = \frac{|H|\,|K|}{|H \cap xKx^{-1}|} = \frac{|H| \cdot |K|}{|x^{-1}Hx \cap K|} \quad ///$$

Sylow Theory is the Converse to Lagrange's Theorem for finite groups.

Say $p^\alpha \| |G|$ if $p^\alpha \mid |G|$ and $p^{\alpha+1} \nmid |G|$.

Let $Syl_p(G) := \{ P \leq G : |P| = p^\alpha \| |G| \}$.

<u>Theorem</u>: If $P \in Syl_p(G)$ and $Q \leq G$ is <u>any</u> p-subgroup then $\exists\, x \in G$ such that $Q \leq xPx^{-1}$. (In particular, all $P \in Syl_p(G)$ are conjugate.)

<u>Proof</u>: Decompose G into double cosets

$$G = \bigsqcup_i Qx_iP$$

$$|G| = \sum_i |Qx_iP| = \sum_i \frac{|Q| \cdot |P|}{|Q \cap x_iPx_i^{-1}|} \quad (*).$$

$\wr$

Suppose for contradiction that

$Q \cap x_i P x_i^{-1} \subsetneq Q \quad \forall i$, hence

If $|Q| = p^r$ then we have

$|Q \cap x_i P x_i^{-1}| = p^s$ with $s < r \quad \forall i$.

$$\Longrightarrow \frac{|Q| \cdot |P|}{|Q \cap x_i P x_i^{-1}|} = \frac{p^{\alpha} \cdot p^{r}}{p^s} = p^{\alpha + r - s}$$

where $\alpha + r - s \geq \alpha + 1 \quad \forall i$.

By (*) this implies $p^{\alpha+1} \mid |G|$. $\times$

$\blacksquare$

Theorem: $Syl_p(G) \neq \emptyset$. In fact, G has subgroups of order p^m $\forall\, m \leq \alpha$.

Proof (Induction on $|G|$):

When G acts on itself by $g * h = g h g^{-1}$ we get a decomposition into conjugacy classes:

$$|G| = |Z(G)| + \sum_{C(x_i) \neq G} |G|/|C(x_i)| \qquad (*)$$

If $p^\alpha \mid |C(x_i)|$ for some $C(x_i) \neq G$ then we're done by induction. Otherwise we use $(*)$ to see that $p \mid |Z(G)|$.

Since $Z(G)$ is abelian this means $\exists\, z \in Z(G)$ with $|\langle z\rangle| = p$ (proof omitted).

Since $\langle z\rangle \trianglelefteq G$ we have a quotient

$$p^{\alpha-1} \mid |G/\langle z\rangle| = |G|/p.$$

By induction $G/\langle z\rangle$ has subgroups of order p^m $\forall\, m \leq \alpha - 1$. Then by Correspondence G has subgroups order p^m $\forall\, m \leq \alpha$.

$$\begin{array}{ccc} G & & G/\langle z\rangle \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \underbrace{H}_{p^{m+1}} & \longleftarrow & \underbrace{H/\langle z\rangle}_{p^m} \end{array} \qquad \blacksquare$$

Remark: This is the best we can do because $|A_4| = 12$ has <u>no</u> subgroup of order $6 = 2 \cdot 3$.

Remarks:

- We also know

$$\underbrace{|Syl_p(G)| = 1 \bmod p}_{\text{don't prove}} \;\&\; \underbrace{|Syl_p(G)| \;\Big|\; \frac{|G|}{p^\alpha}}_{\text{prove.}}$$

- Simplicity of PSL NOT on the exam

- Linear Representations (Schur & Maschke, etc.) are NOT on the exam